The Effect of Dissipation on the Transformation-Based Approximate Electromagnetic Cloaking Scheme

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Math 5440

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Outline

1. Helmholtz Equation
   - In 2D, models the propagation of TE and TM waves
   - In 3D, models the propagation of acoustic waves
2. Cloaking for the 2D Helmholtz Equation
   - Boundary Measurements Map
   - Definition of Cloaking
   - Main Idea Behind Transformation-Based Cloaking
   - Dissipation and Results

Wave Equation

Maxwell's Equations imply that the electric field satisfies the wave equation, namely

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Assuming the waves are traveling in the $y-$direction, the solution is of the following form:

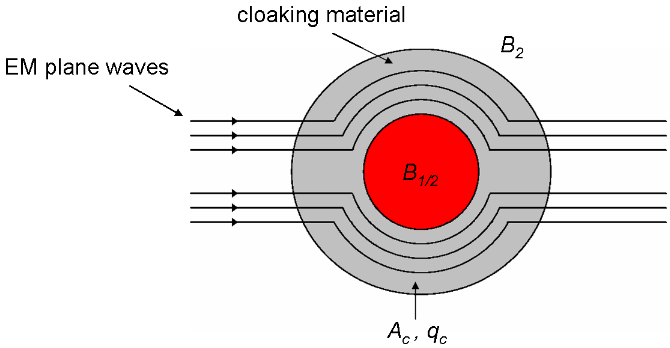
$$\mathbf{E}(y,t) = \mathrm{Re}\left[\widetilde{\mathbf{E}}_0 e^{i(ky-\omega t)}\right]$$

Helmholtz Equation

$$\nabla \cdot [A(\mathbf{x}) \nabla u(\mathbf{x})] + \omega^2 q(\mathbf{x}) u(\mathbf{x}) = 0$$

1. When derived from Maxwell's Equations, $A(\mathbf{x})$ and $q(\mathbf{x})$ are related to $\mu(\mathbf{x})$ and $\epsilon(\mathbf{x})$.
2. u represents the longitudinal component of the electric field (TM waves) or the magnetic field (TE waves).
3. In 2D, it models the propagation of monochromatic electromagnetic transverse plane waves.
4. In 3D, it models the propagation of acoustic waves.

What is Cloaking?



1. Assume radial symmetry
2. Cloak against plane waves

Boundary Measurements Map

$$\nabla \cdot [A(\mathbf{x})\nabla u(\mathbf{x})] + \omega^2 q(\mathbf{x})u(\mathbf{x}) = 0$$

1. We consider B_2.
2. Maps field to flux for the Helmholtz equation.
3. We used the inverse of this map (more stable numerically).

Definition (Boundary Measurements Map)

The boundary measurements map associated with the Helmholtz equation is defined by:

$$\Lambda_{A,q}(u) = (A\nabla u) \cdot \nu \,, \tag{1}$$

where ν is the outward normal vector of B_2, and $\nu(\mathbf{x}) = \dfrac{\mathbf{x}}{2}$ for $\mathbf{x} \in \partial B_2$.

Invariance Principle

$$\nabla_x \cdot [A(\mathbf{x})\nabla_x u(\mathbf{x})] + \omega^2 q(\mathbf{x})u(\mathbf{x}) = 0$$

$\mathbf{F} : B2 \rightarrow B2$, smooth enough, $\mathbf{F}(\mathbf{x}) = \mathbf{x}$ on ∂B_2.

Changing variables with $\mathbf{y} = \mathbf{F}(\mathbf{x})$ gives

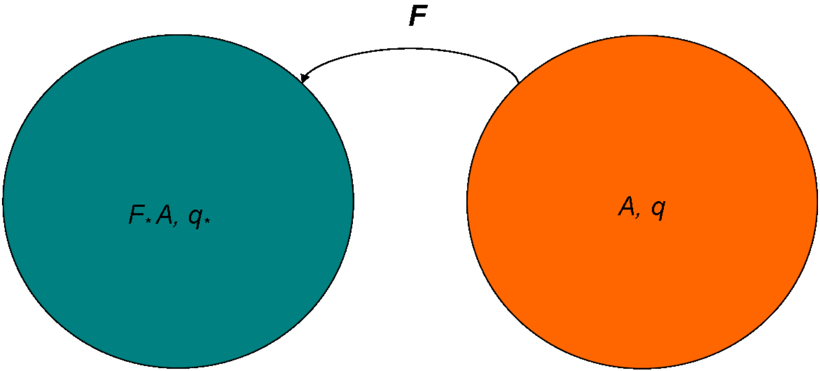
$$\nabla_y \cdot [F_* A(\mathbf{y})\nabla_y v(\mathbf{y})] + \omega^2 q_*(\mathbf{y})v(\mathbf{y}) = 0\,,$$

where $v(\mathbf{y}) = u[\mathbf{F}^{-1}(\mathbf{y})]$.

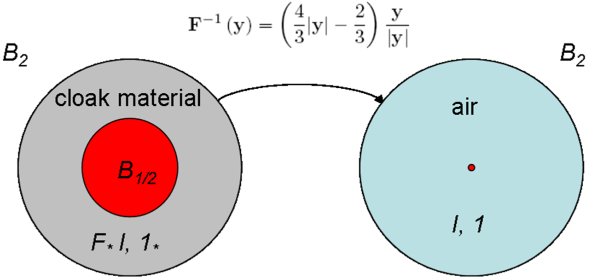
Also,

$$\Lambda_{A,\,q} = \Lambda_{F_* A,\,q_*}$$

Invariance Principle



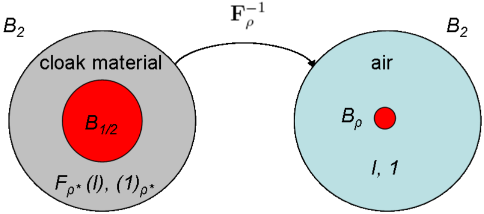
Transformation-Based Cloaking Proposed by Pendry



$$\Lambda_{F_*I,1_*} = \Lambda_{I,1}$$

$$A(\mathbf{x}), q(\mathbf{x}) = \begin{cases} F_*I, 1_* & \text{in } B_2 \setminus B_{1/2} \\ \text{arbitrary real} & \text{in } B_{1/2} \end{cases}$$

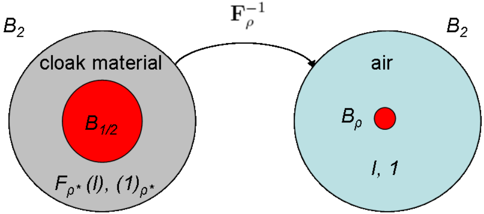
Transformation-Based Approximate Cloaking Scheme



$$\Lambda_{A,q} = \Lambda_{A_\rho, q_\rho}$$

$$\mathbf{F}_\rho^{-1}(\mathbf{y}) = \begin{cases} 2\rho\,\mathbf{y} & \text{for } |\mathbf{y}| < \frac{1}{2} \\ \left[2 - \frac{4\,(2-\rho)}{3} + \frac{2\,(2-\rho)}{3}|\mathbf{y}|\right]\frac{\mathbf{y}}{|\mathbf{y}|} & \text{for } \frac{1}{2} \le |\mathbf{y}| \le 2\,. \end{cases}$$

Transformation-Based Approximate Cloaking Scheme



$$\Lambda_{A,q} = \Lambda_{A_\rho, q_\rho} \neq \Lambda_{I,1}$$

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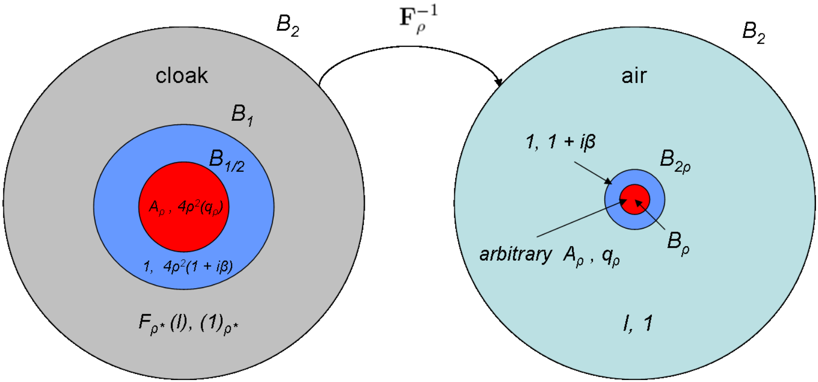
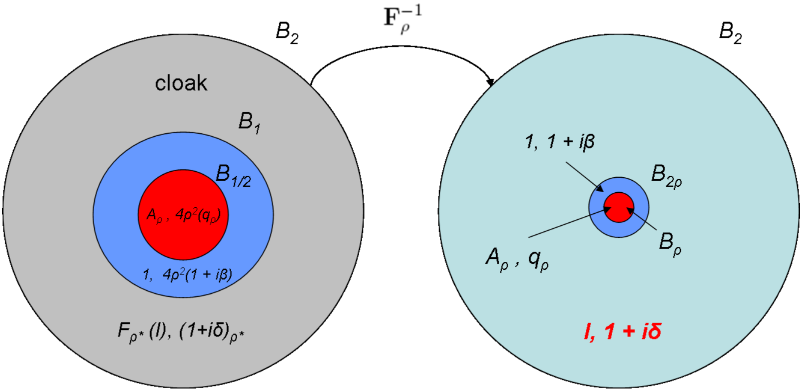
Kohn, Onofrei, Vogelius, and Weinstein

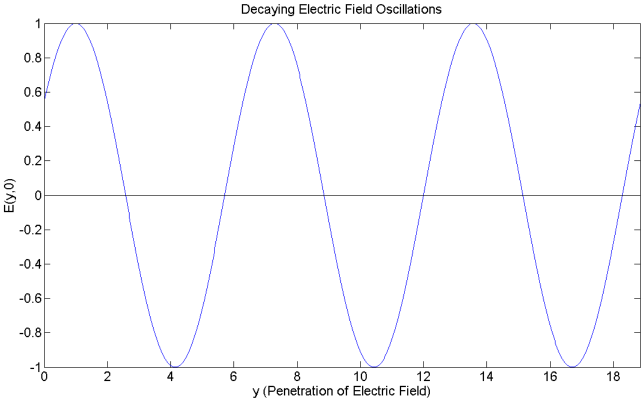


1. In 2D, error $\sim \dfrac{1}{|\log(\rho)|}$.
2. In 3D, error $\sim \rho$.

Dissipation



Effect of Dissipation



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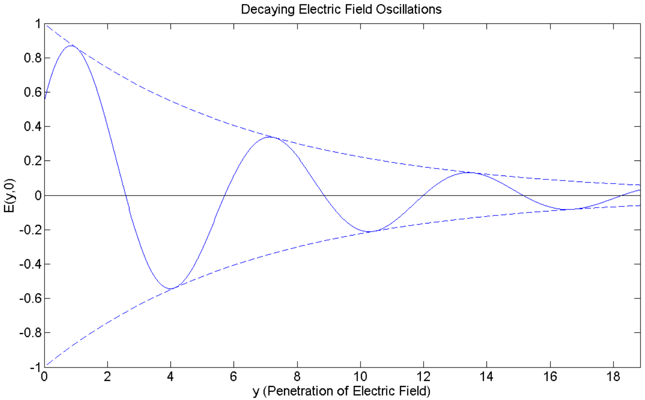
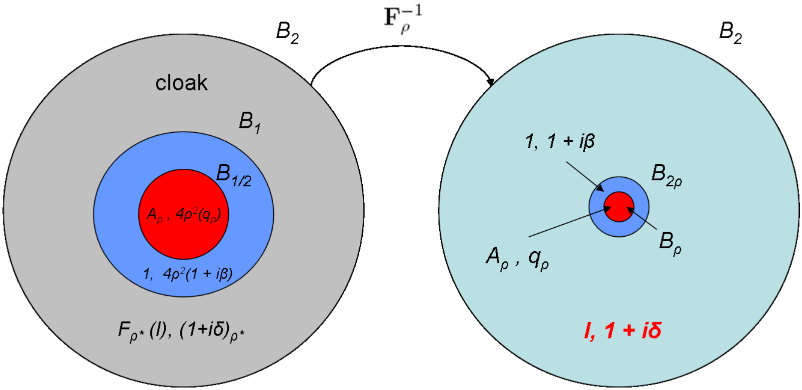
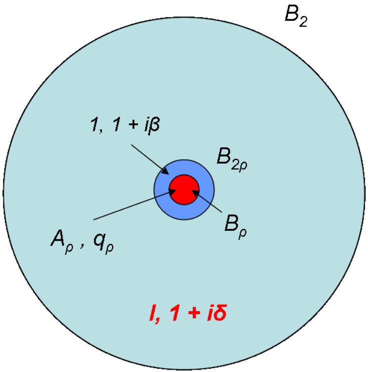


Step 1–Mapping



Step 2–Setup



Step 2–Setup

$$
\begin{cases}
\text{(a) } \nabla^2 u_1(\mathbf{x}) + \omega^2 \dfrac{q_\rho}{A_\rho} u_1(\mathbf{x}) = 0 & \text{in } B_\rho \\
\text{(b) } \nabla^2 u_2(\mathbf{x}) + \omega^2 (1 + i\beta) u_2(\mathbf{x}) = 0 & \text{in } B_{2\rho} \setminus B_\rho \\
\text{(c) } \nabla^2 u_3(\mathbf{x}) + \omega^2 (1 + i\delta) u_3(\mathbf{x}) = 0 & \text{in } B_2 \setminus B_{2\rho} \\
\text{(d) } u_1(\rho,\theta) = u_2(\rho,\theta); \quad A_\rho \dfrac{\partial u_1}{\partial r}(\rho,\theta) = \dfrac{\partial u_2}{\partial r}(\rho,\theta) & \text{on } \partial B_\rho \\
\text{(e) } u_2(2\rho,\theta) = u_3(2\rho,\theta); \quad \dfrac{\partial u_2}{\partial r}(2\rho,\theta) = \dfrac{\partial u_3}{\partial r}(2\rho,\theta) & \text{on } \partial B_{2\rho} \\
\text{(f) } \dfrac{\partial u_3}{\partial r}(2,\theta) = \psi & \text{on } \partial B_2
\end{cases}
$$

Step 2–Setup

where

$$q_\rho = \frac{z_e J_0(\omega\rho)\left[J_0'(\omega z_e \rho)\left(H_0^{(1)}\right)'(2\omega z_e) - J_0'(2\omega z_e)\left(H_0^{(1)}\right)'(\omega z_e \rho)\right]}{J_0'(\omega\rho)\left[J_0(\omega z_e \rho)\left(H_0^{(1)}\right)'(2\omega z_e) - J_0'(2\omega z_e) H_0^{(1)}(\omega z_e \rho)\right]}$$

$$A_\rho = q_\rho I$$

Step 2–Setup

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- $z_e = 1 + i\delta$

Step 2–Setup

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- $\delta > 0$ denotes the dissipation level in the annulus $B_2 \setminus B_{2\rho}$

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$$A_\rho = q_\rho I$$

- $z_e = 1 + i\delta$
- $\delta > 0$ denotes the dissipation level in the annulus $B_2 \setminus B_{2\rho}$
- ψ is the normal derivative of the incoming plane wave on ∂B_2

Separation of Variables for u_1

Let $u_1(r,\theta) = R(r)\Theta(\theta)$. Then we have

$$\nabla^2 u_1 + \omega^2 \frac{q_\rho}{A_\rho} u_1 = 0$$

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$$\Leftrightarrow \frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_1}{\partial \theta^2} + \omega^2 \frac{q_\rho}{A_\rho} u_1 = 0$$

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$$\Leftrightarrow r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} + \omega^2 \frac{q_\rho}{A_\rho} r^2 = 0$$

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$$\begin{aligned}
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\Leftrightarrow\ & \frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_1}{\partial \theta^2} + \omega^2 \frac{q_\rho}{A_\rho} u_1 = 0 \\
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\Leftrightarrow\ & r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} + \omega^2 \frac{q_\rho}{A_\rho} r^2 = 0 \\
\Leftrightarrow\ & r^2\frac{R''}{R} + r\frac{R'}{R} + \omega^2 \frac{q_\rho}{A_\rho} r^2 = -\frac{\Theta''}{\Theta} = K\,.
\end{aligned}$$

3 Cases

- $K < 0$ *e.g.* $K = -\mu^2$.

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- $K > 0$ *e.g.* $K = \mu^2$.
  - $\Theta'' + \mu^2\Theta = 0$.
  - $\Theta(\theta) = Ee^{i\mu\theta} + Fe^{-i\mu\theta}$.
  - $2\pi-$periodic iff $\mu = k$, where k is a non-negative integer

Bessel's Equation

The equation for R becomes

$$r^2 R_k'' + rR_k' + \left(\omega^2 \frac{q_\rho}{A_\rho} r^2 - k^2\right) R = 0\,.$$

(The boundary conditions will be dealt with momentarily). The solution to Bessel's Equation of order k is

$$R_k(r) = P_k J_k\left(\omega\sqrt{\frac{q_\rho}{A_\rho}}r\right) + Q_k H_k^{(1)}\left(\omega\sqrt{\frac{q_\rho}{A_\rho}}r\right)$$

Superposition

Thus

$$\begin{aligned} u_{1,k}(r,\theta) &= R_k(r)\Theta_k(\theta) \\ &= \left[P_k J_k\left(\omega\sqrt{\frac{q_\rho}{A_\rho}}r\right) + Q_k H_k^{(1)}\left(\omega\sqrt{\frac{q_\rho}{A_\rho}}r\right)\right]\left(Ee^{ik\theta} + Fe^{-ik\theta}\right) . \end{aligned}$$

Superposition

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$$= \left[P_k J_k\left(\omega\sqrt{\frac{q_\rho}{A_\rho}}r\right) + Q_k H_k^{(1)}\left(\omega\sqrt{\frac{q_\rho}{A_\rho}}r\right)\right]\left(Ee^{ik\theta} + Fe^{-ik\theta}\right).$$

Superposing these solutions gives

$$u_1(r,\theta) = \sum_{k=0}^{\infty}\left[a_k J_k\left(\omega\sqrt{\frac{q_\rho}{A_\rho}}r\right) + b_k H_k^{(1)}\left(\omega\sqrt{\frac{q_\rho}{A_\rho}}r\right)\right]\left(e^{ik\theta} + e^{-ik\theta}\right)$$
$$= \sum_{k=-\infty}^{\infty}\left[a_k J_k\left(\omega\sqrt{\frac{q_\rho}{A_\rho}}r\right) + b_k H_k^{(1)}\left(\omega\sqrt{\frac{q_\rho}{A_\rho}}r\right)\right]e^{ik\theta}.$$

Solution

$$\begin{cases} \text{(a) } u_1(r,\theta) = \sum_{k=-\infty}^{\infty} a_k J_k\left(\sqrt{\frac{q_\rho}{A_\rho}}\omega r\right) e^{ik\theta} & \text{in } B_\rho \\ \text{(b) } u_2(r,\theta) = \sum_{k=-\infty}^{\infty} \left[c_k J_k\left(\omega z_b r\right) + d_k H_k^{(1)}\left(\omega z_b r\right)\right] e^{ik\theta} & \text{in } B_{2\rho} \setminus B_\rho \\ \text{(c) } u_3(r,\theta) = \sum_{k=-\infty}^{\infty} \left[e_k J_k\left(\omega z_e r\right) + f_k H_k^{(1)}\left(\omega z_e r\right)\right] e^{ik\theta} & \text{in } B_2 \setminus B_{2\rho} \end{cases}$$

Error

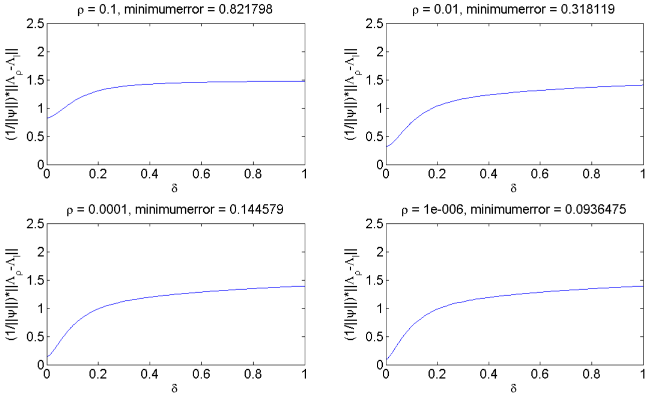
The error we measured was the following normalized error:

$$\frac{\left(\int_{\partial B_2} |u_3 - u_0|^2\right)^{\frac{1}{2}}}{\left(\int_{\partial B_2} |\psi|^2\right)^{\frac{1}{2}}}$$

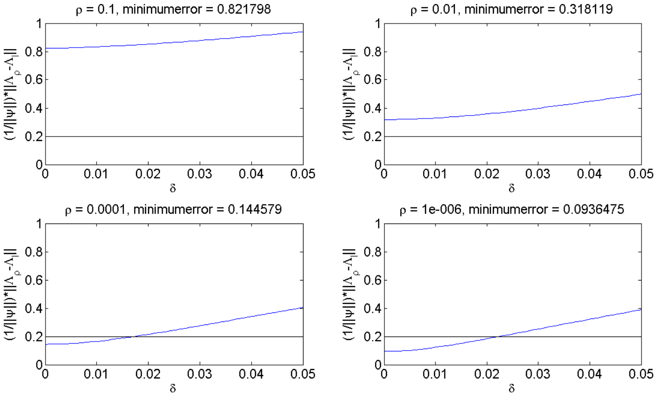
or, equivalently,

$$\frac{\left[\int_{\theta=0}^{2\pi} |u_3(2,\theta) - u_0(2,\theta)|^2 \, \mathrm{d}\theta\right]^{\frac{1}{2}}}{\left[\int_{\theta=0}^{2\pi} |\psi(2,\theta)|^2 \, \mathrm{d}\theta\right]^{\frac{1}{2}}}$$

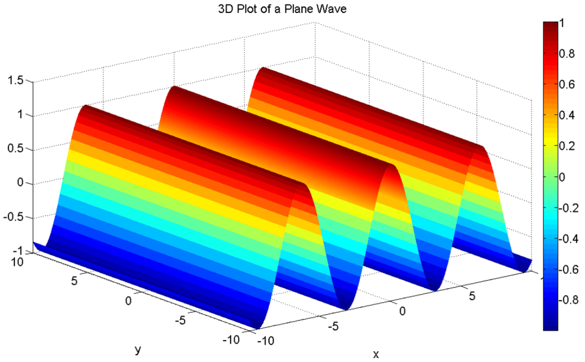
Dependence on ρ and δ



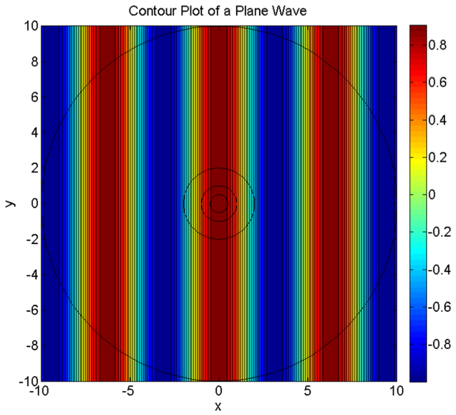
Dependence on ρ and δ



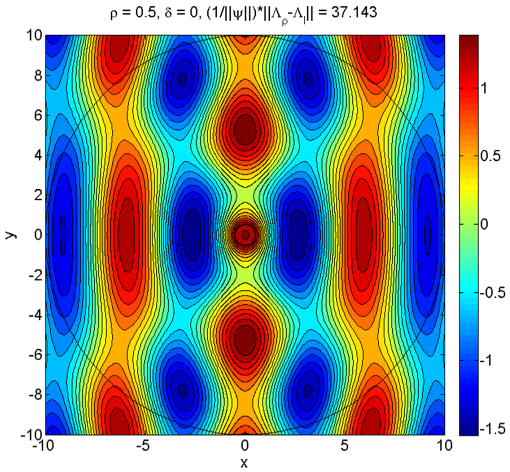
Electromagnetic Wave



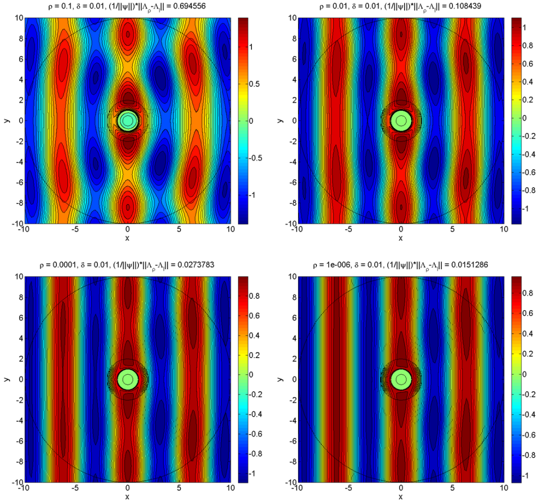
Contour Plot of Electromagnetic Wave



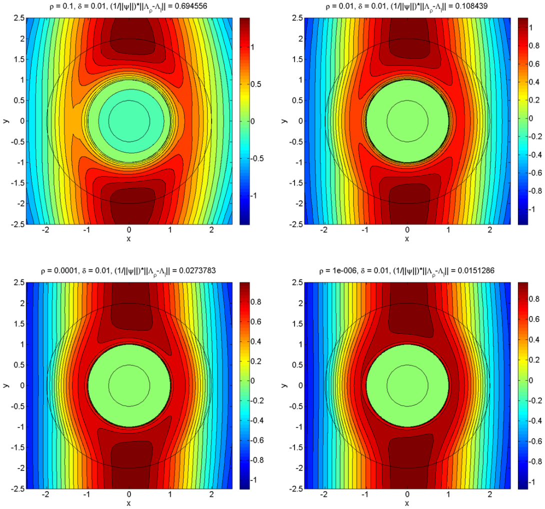
Shadow



Cloaking


ρ = 0.1, δ = 0.01, (1/||ψ||)*||Λρ-ΛI|| = 0.694556
ρ = 0.01, δ = 0.01, (1/||ψ||)*||Λρ-ΛI|| = 0.108439
ρ = 0.0001, δ = 0.01, (1/||ψ||)*||Λρ-ΛI|| = 0.0273783
ρ = 1e-006, δ = 0.01, (1/||ψ||)*||Λρ-ΛI|| = 0.0151286
x
y

Zoom-in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x
y

Bessel's Equation of Order 0

$$R''(r) + \frac{1}{r}R'(r) + R(r) = 0$$

We assume a series solution of the form

$$R(r) = \sum_{n=0}^{\infty} c_n r^n$$

Assuming we can differentiate term-by-term, we have

$$R'(r) = \sum_{n=1}^{\infty} n c_n r^{n-1}$$

$$R''(r) = \sum_{n=2}^{\infty} n(n-1) c_n r^{n-2}$$

Bessel's Equation of Order 0

Inserting the series into Bessel's equation gives

$$\sum_{n=2}^{\infty} n(n-1)c_n r^{n-2} + \frac{1}{r}\sum_{n=1}^{\infty} nc_n r^{n-1} + \sum_{n=0}^{\infty} c_n r^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n r^{n-2} + \sum_{n=1}^{\infty} nc_n r^{n-2} + \sum_{n=0}^{\infty} c_n r^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n r^{n-2} + c_1 r^{-1} + \sum_{n=2}^{\infty} nc_n r^{n-2} + \sum_{n=2}^{\infty} c_{n-2} r^{n-2} = 0$$

$$\frac{c_1}{r} + \sum_{n=2}^{\infty} \left(n^2 c_n + c_{n-2}\right) r^{n-2} = 0$$

Bessel's Equation of Order 0

This implies that

$$c_1 = 0$$
$$c_n = -\frac{c_{n-2}}{n^2}$$

Using induction, we find that the solution to the above recurrence relation is

$$c_n = \begin{cases} 0 \text{ for } n \text{ odd} \\ \frac{(-1)^k c_0}{2^{2k} (k!)^2} \text{ for } n \text{ even, where } k = \frac{n}{2} \end{cases}$$

Bessel's Equation of Order 0

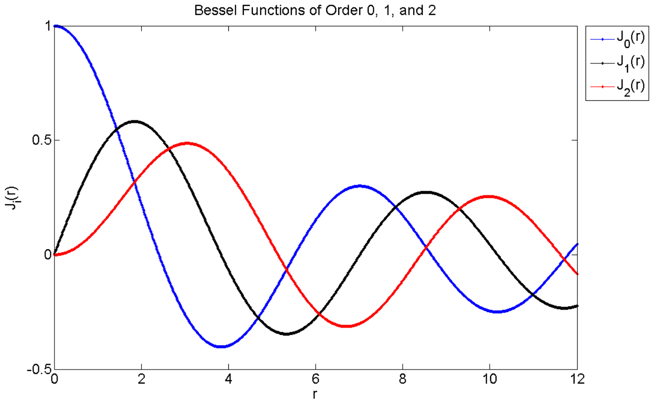
Thus our solution is

$$\begin{aligned} R(r) &= \sum_{k=0}^{\infty} \frac{(-1)^k c_0}{2^{2k} (k!)^2} r^{2k} \\ &= c_0 + \sum_{k=0}^{\infty} \frac{(-1)^k c_0}{2^{2k} (k!)^2} r^{2k} \end{aligned}$$

The ratio test shows that the above series converges for all r.
We thus define

$$J_0(r) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} (k!)^2} r^{2k}$$

Bessel's Equation of Orders 0, 1, 2, and 10



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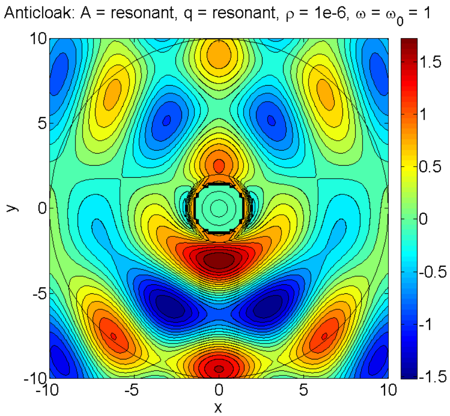
Anticloak: A = resonant, q = resonant, ρ = 1e-6, ω = ω0 = 1
10
5
0
-5
-10
y
-10
-5
0
5
10
x
1.5
1
0.5
0
-0.5
-1
-1.5

Thank you!!!
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