Math 5210
Friday March 6

next hw assignment will be due the Friday after spring break
– you should be digesting course material for exam on Wed!

- Finish Stone-Weierstrass

- Midterm is in class next week – Wed 3/11.
  Based primarily on class notes/discussion & hw.

$\boxed{\text{Real numbers}}$ 2.1–2.3, 3.1 1/12 – 1/21

not on exam: the construction details of $\mathbb{R}$ as the completion of $\mathbb{Q}$ (although the later theorem that any metric space can be completed is important.)

← On exam you can use that $\mathbb{R}$ is a complete Archimedean field

possible: defs of Cauchy, convergence,
limit theorems, e.g. $\{x_k\} \to x$, $\{y_k\} \to y \Rightarrow \{x_k y_k\} \to xy$
$\{x_k + y_k\} \to x + y$
$\{1/x_k\} \to 1/x$ if $x \neq 0$.
be able to work with ε's, δ's, $\frac{1}{n}$'s, M's, etc.

limit pts of sequences, sups, infs, limsup's, liminf's (def's & $\exists$)
→ "divide and conquer" algorithm

$\boxed{\text{General metric spaces}}$ 9.1–9.3 1/23 – 2/13

not on exam: proof of $\|\cdot\|_p$ Δ ineq, except $p = 1, 2, \infty$ are possible
Hausdorff distance results.
Arzela-Ascoli (about cpt subsets of $(C([a,b]), \|\cdot\|_\infty)$)

possible:
metric space, Cauchy, convergent seq's, complete metric space
normed vector spaces (Banach if complete), inner product spaces (Hilbert if complete)
(how to get the norm, C.S., Δ ineq.)
why $\mathbb{R}^n$ is complete (for any & all p-norms)
equivalent metrics
→ why $(C([a,b]), \|\cdot\|_\infty)$ is complete (& generalizations)
open & closed subsets, interior, closure, complement
metric subspaces (& relatively open & closed sets)
→ compactness (3 equiv. characterizations)
continuity (3 equiv def's)
connectivity, path connectivity, path connected $\Rightarrow$ connected.
$X \subset \mathbb{R}$ connected iff X = interval
→ Theorems about cont fcns, e.g.
$f, g : (X, d) \to \mathbb{R}$ cont $\Rightarrow f+g,\ fg,\ f/g\ (g \neq 0)$ cont
$f : (X, d) \to \mathbb{R}^n$ cont iff each comp. fcn f_i is
compositions
$f : X \to Y$ cont, X cpt $\Rightarrow f(X)$ cpt
" conn $\Rightarrow$ " conn
path conn $\Rightarrow$ " path conn.
$f : X \to \mathbb{R}$ cont, X cpt $\Rightarrow f$ attains inf & sup
$f : X \to Y$ cont, X cpt $\Rightarrow f$ uniformly cont
$d(x, x_0)$, $x \in X$ cont, $d(x, y)$, $(x,y) \in X \times Y$ cont
→ Contraction mapping principle

Integration and differentiation

2/17 – 3/6

book §6.1 (except our def $\int_a^b f(t)dt$ is better)
10.1, 11.1–11.4, parts we talked about
7.3–7.5

→ $\begin{cases} f:[a,b]\to X_{\text{Banach}},\ f \text{ cont} \Rightarrow \exists \int_a^b f(x)dx = \lim_{\|P\|\to 0} \mathcal{R}_P(f) \\ \quad \text{properties of the def. integral} \\ \text{FTC (a), (b)} \end{cases}$

$f:[a,b]\to X_{\text{Banach}}$ differentiable

→ $\begin{cases} f: X_{\text{Ban}} \to Y_{\text{Ban}} \text{ differentiable at } x\in X \\ \text{operator norm} \\ \text{chain rule} \end{cases}$

→ local $\exists!$ thm for IVP for 1st order systems via contraction mapping princ.
also global results

→ interchanging limits of fcns with derivative & integrals
(Thms 1 & 2 Feb 27)

power series
convolution, approximate identities
Weirestrass approx thm
Stone-Weirstrass thm (9.3)

arrows indicate current favorites, but all is possible
typical problems:
define something
prove something (or find counterexample)
could be short basic thm, or relatively straightforward HW-like,
or one of the fundamental more complicated results.

An application of our approximate identity construction: (from PDE's)

<u>Theorem</u>: Let IVP $\begin{cases} u_t = k u_{xx} & t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \end{cases}$

be the initial value problem for the heat equation in one space variable.

(or diffusion)

Assume $f(x)$ is continuous, and $f(x) \to 0$ as $|x| \to \infty$.

Then the <u>unique</u> soltn to IVP is given by

(among soltns $u(x,t)$ which $\to 0$ as $|x| \to \infty$, for fixed t)

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy = (f * g_{2\sqrt{t}})(x)$$

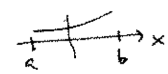
for $g(z) = \frac{1}{\sqrt{\pi}} e^{-z^2}$

$g_\lambda(z) = \frac{1}{\lambda} g\left(\frac{z}{\lambda}\right)$

as in previous notes.

<u>sketch of proof</u>

(0) <u>derivation of PDE</u>: in diffusion problems it is assumed that over any test interval $[a,b] \subset \mathbb{R}$,



$$\frac{d}{dt} \underbrace{\int_a^b u(x,t)\, dx}_{\text{total "heat" in interval}} = k\,(u_x(b) - u_x(a))$$

k: diffusivity constant

i.e. heat content changes only by heat flowing in thru endpoints, at rates proportional to u_x.

after a suitable interchange of $\frac{d}{dt}$ & integral,

which can be justified under appropriate hypotheses, like "Thm 1" a few days ago

$$\int_a^b u_t(x,t)\, dx = k \int_a^b u_{xx}\, dx$$

i.e.

$$\int_a^b u_t - k u_{xx}\, dx = 0$$

$\forall\, [a,b]$,

so if u_t, u_{xx} are continuous,

FTC $\Rightarrow$ $\boxed{u_t - k u_{xx} = 0}$

$$\left(\frac{d}{dz} \int_a^z u_t - k u_{xx}\, dx = 0 \right)$$

(1) uniqueness : If u,v solve IVP and $\to 0$ for fixed t, as $x \to \infty$, then

$$\frac{d}{dt}\int_{\mathbb{R}} (u-v)^2\,dx = \int_{\mathbb{R}} \frac{d}{dt}(u-v)^2\,dx = \int_{\mathbb{R}} 2(u-v)(u-v)_t\,dx = 2k\int_{\mathbb{R}} (u-v)(u-v)_{xx}\,dx$$

$$= \underbrace{2k\left((u-v)(u-v)_x\right]_{-\infty}^{\infty}}_{=0 \text{ by hyp.}} \underbrace{-2k\int_{\mathbb{R}} (u-v)_x(u-v)_x\,dx}_{\leq 0}$$

thus if $u=v$ initially,
then $u=v$ $\forall\, t>0$, since

$$\int_{\mathbb{R}} (u-v)^2(x,t)\,dx \searrow \text{ in } t\,!$$

(2) $u(x,t) = (f * g_{2\sqrt{t}})(x)$, so by our previous work (modified to include $f \to 0$ as $|x| \to \infty$, rather than f has compact sppt.)

$$f * g_{2\sqrt{t}} \to f \quad \text{uniformly as } t \searrow 0.$$

this implies $\tilde{u}(x,t) := \begin{cases} u(x,t) & t>0 \\ f(x) & t=0 \end{cases}$

is continuous in the half plane $t \geq 0$, $x \in \mathbb{R}$
(you could check this.)

thus our $u(x,t)$ has ~~since~~ the correct initial values

(3) $u(x,t)$ solves the PDE!

One can justify (but it's easiest with the Lebesgue integral and the dominated convergence theorem, since $\mathbb{R}$ is not a bounded interval and our unif. conv. thm won't work)

$$(\partial_t - \partial_x\partial_x)\,\frac{1}{2\sqrt{\pi k}}\int_{y=-\infty}^{\infty} \frac{1}{\sqrt{t}}\, e^{-(x-y)^2/4kt} f(y)\,dy$$

$$= \frac{1}{2\sqrt{\pi k}}\int_{y=-\infty}^{\infty} \Big\{ \underbrace{(\partial_t - \partial_x\partial_x)\left[\frac{1}{\sqrt{t}}\, e^{-(x-y)^2/4kt}\right]}_{\text{it is just a derivative calculation to show this equals zero (messy).}} \Big\} f(y)\,dy$$

(all this works for $\vec{x} \in \mathbb{R}^2$ or $\mathbb{R}^3$ too.)

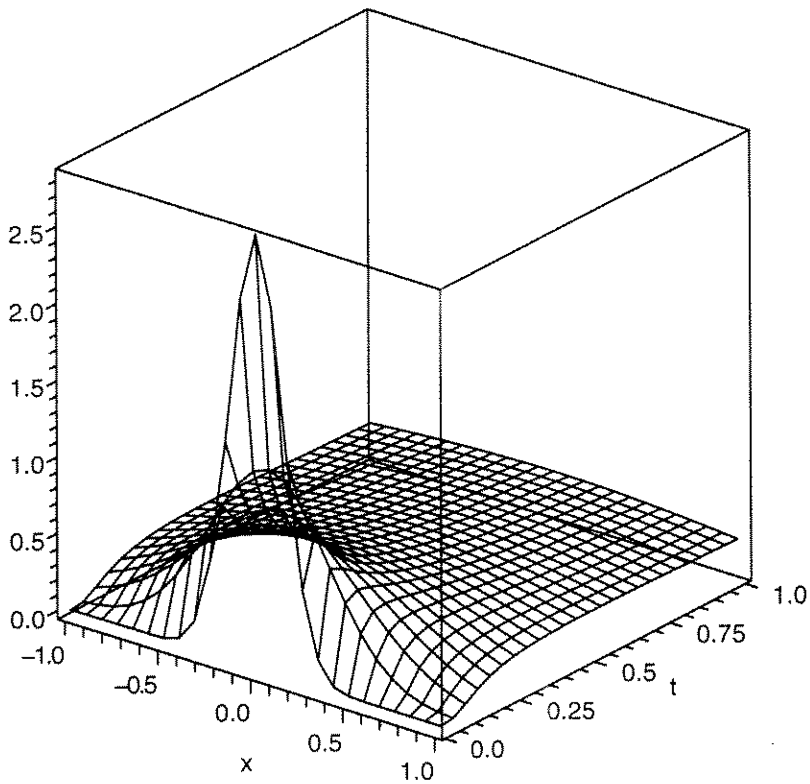
Soln is product of these solns,
in each variable x_1, x_2, x_3 etc.

> $with(plots):$

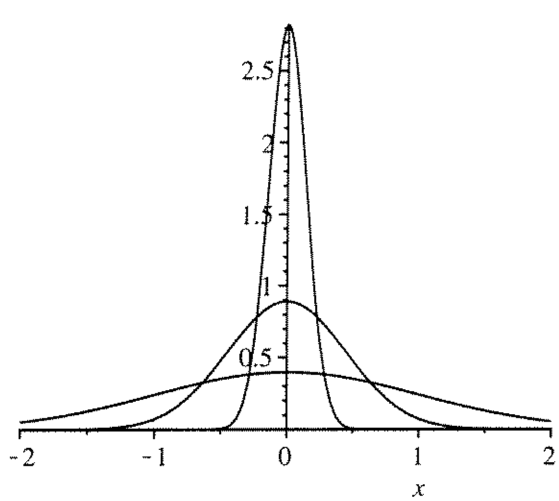
> $g := (x, t) \rightarrow \frac{1}{(2 \cdot \mathrm{sqrt}(\mathrm{Pi} \cdot t))} \cdot \exp\left(-\frac{x^2}{(4 \cdot t)}\right);$

$$g := (x, t) \rightarrow \frac{1}{2} \frac{e^{-\frac{1}{4} \frac{x^2}{t}}}{\sqrt{\pi t}} \qquad \textbf{(1)}$$

> $plot3d(g(x, t), x = -1..1, t = .01..1, color = black, style = wireframe, axes = boxed);$



> $plot(\{ g(x, .1), g(x, .5), g(x, .01) \}, x = -2..2, color = black);$



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