Math 5210
March 4 Wed.

First: Finish Weierstrass thm: $f:[a,b]\to\mathbb{R}$ cont, $\varepsilon>0$

$\Rightarrow \exists\ p_n(x)$ poly, $\|f-p_n\|_{sup}<\varepsilon$, from Tuesday

A strong generalization of this deals with more general situation:

X compact space
$C(X,\mathbb{R}) := \{f: X\to\mathbb{R} \text{ cont}\}$.

an algebra is a vector space, together with a multiplication satisfying associative, distributive laws

examples : $C(X,\mathbb{R})$, with $(fg)(x) := f(x)g(x)$
$(c_1 f + c_2 g)(x) := c_1 f(x) + c_2 g(x)$.

$M_{n\times n} = \{n\times n \text{ matrices}\}$.

Stone-Weierstrass approx thm (§9.3) ~ we skipped this section originally.

Let X compact metric space
Consider $(C(X,\mathbb{R}), \|\cdot\|_{sup})$
Let $\mathcal{A}\subset C(X,\mathbb{R})$ ~~which~~ be a subalgebra with the additional hypotheses

$\left.\begin{array}{l}\text{(i) } \forall x\in X\ \exists\, g\in\mathcal{A} \text{ s.t. } g(x)\neq 0\\ \text{(ii) } \forall\, x,y\in X\ \exists\, g\in\mathcal{A} \text{ s.t. } g(x)\neq g(y)\end{array}\right\}$ $\mathcal{A}$ "strongly separates points"

Then $\overline{\mathcal{A}} = C(X,\mathbb{R})$
i.e. $\forall\varepsilon>0,\ f\in C(X,\mathbb{R})\ \exists\, g\in\mathcal{A}$ s.t. $\|f-g\|_{sup}<\varepsilon$.

examples

- $C([a,b])$, and the polynomial subalgebra : i.e. the Weierstrass approx thm (which we use to prove Stone-W.).
  → notice affine fcns $p_1(x) = cx+d$ ~~ax+b~~ already strongly separate points
- $X\subset\mathbb{R}^n$, X closed and bounded, $C(X,\mathbb{R})$.
  $\mathcal{A} = \{\text{polynomials in } x_1,\dots x_n\}$
  → notice then coord fcns $p_j(\vec{x}) = x_j$, and constants, already strongly separate points.
- $S^1 = \{\binom{x}{y}\in\mathbb{R}^2 \text{ s.t. } x^2+y^2+1\}$
  $= \left\{\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\right\}$.

  finite lin. combos

  $C(S^1,\mathbb{R})$. $\mathcal{A} = \text{span}\{1, \cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \cos 3\theta, \sin 3\theta, \dots\}$

  separates pts: span$(\cos\theta, \sin\theta)$ already does
  this is an algebra : use trig identities (or rederive, using

  $\cos n\theta = \frac{1}{2}(e^{in\theta}+e^{-in\theta})$
  $\sin n\theta = \frac{1}{2i}(e^{in\theta}-e^{-in\theta})$)

proof of Stone-Weierstrass : Preliminaries :

Lemma 1 $\forall\ x, y \in X$ distinct $\exists\ g \in \mathcal{A}$ $\quad g(x) = c_1$
$c_1, c_2 \in \mathbb{R}$ $\quad g(y) = c_2$

proof pick $g_1 \in \mathcal{A}$, $g_1(x) \neq g_1(y)$

- if both values are non-zero, then the matrix $\begin{bmatrix} g_1(x) & g_1^2(x) \\ g_1(y) & g_1^2(y) \end{bmatrix}$ is nonsingular (e.g. check det.)

so solve $\begin{bmatrix} g_1(x) & g_1^2(x) \\ g_1(y) & g_1^2(y) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ for $\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$

and set $g = t_1 g_1 + t_2 g_1^2 \in \mathcal{A}$.

- if one value, say $g_1(x)$, equals zero, pick g_2 s.t. $g_2(x) \neq 0$. Then the matrix $\begin{bmatrix} g_1(x) & g_2(x) \\ g_1(y) & g_2(y) \end{bmatrix}$ is nonsing (det ≠ 0!)

and solve $\begin{bmatrix} g_1(x) & g_2(x) \\ g_1(y) & g_2(y) \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, set $g = t_1 g_1 + t_2 g_2 \in \mathcal{A}$ ∎

Lemma 2 $g \in \bar{\mathcal{A}} \Rightarrow |g| \in \bar{\mathcal{A}}$

(this is where we use Weierstrass for $C([a,b])$)

Let $\varepsilon > 0$, $g \in \bar{\mathcal{A}}$. Pick $g_1 \in \mathcal{A}$, $\|g - g_1\|_{sup} < \varepsilon/2$.

X compact $\Rightarrow g_1(X) \subset [-M, M]$ some M.

pick a polynomial $p_n(t)$ s.t. $\| |t| - p_n(t) \|_{\text{sup on } [-M,M]} < \varepsilon/2$

$\Rightarrow \| |g_1| - p_n \circ g_1 \|_{\text{sup on } X} < \varepsilon/2$

now $p_n \circ g_1 \in \mathcal{A}$ since $\mathcal{A}$ is an algebra.

and $\| |g| - p_n \circ g_1 \|_{sup} \leq \| |g| - |g_1| \|_{sup} + \| |g_1| - p_n \circ g_1 \|_{sup}$
$< \varepsilon/2 + \varepsilon/2$ ∎

note $\| |g| - |g_1| \|_{sup} \leq \| g - g_1 \|_{sup}$ since $| |a| - |b| | \leq |a - b|$.

Lemma 3 $\bar{\mathcal{A}}$ is an algebra

pf: if $\{g_n\} \subset A$, $\{g_n\} \to g \in \bar{\mathcal{A}}$ $(\|\cdot\|_{sup})$
$\{h_n\} \subset A$, $\{h_n\} \to h \in \bar{\mathcal{A}}$

then $\{g_n h_n\} \to gh \Rightarrow gh \in \bar{\mathcal{A}}$
$c_1 g_n + c_2 h_n \to c_1 g + c_2 h \in \bar{\mathcal{A}}$. ∎

Lemma 4: $g, h \in \bar{\mathcal{A}} \Rightarrow \max(g,h) = \frac{g+h}{2} + \frac{|g-h|}{2} \in \bar{\mathcal{A}}$ (Lemmas 2,3)

$\min(g,h) = \frac{g+h}{2} - \frac{|g-h|}{2} \in \bar{\mathcal{A}}$

so also the max (or min) of any finite collection of fns in $\bar{\mathcal{A}}$ is in $\bar{\mathcal{A}}$.

proof of S-W.

Let $f \in C(X, \mathbb{R})$.

suffices to show that $\forall \varepsilon > 0 \ \exists g \in \bar{\mathcal{A}}$ s.t. $\|f-g\|_{sup} \leq \varepsilon$.

Let $\varepsilon > 0$.

<u>Step 1</u> for each $z \in X$ we construct $g_z(x) \in \bar{\mathcal{A}}$ s.t

$$\begin{cases} g_z(z) = f(z) \\ g_z(x) \geq f(x) - \varepsilon \quad \forall x \in X. \end{cases}$$

how: $\forall y \in X$

$\exists \ g_{yz} \in \mathcal{A}$ s.t.

$g_{yz}(y) = f(y)$

$g_{yz}(z) = f(z)$ (Lemma 1) also $y=z$ o.k. since $\mathcal{A}$ s.s.pts.

$\Rightarrow \exists \ r_y$ s.t. $g_{yz}(x) \geq f(x) - \varepsilon \quad \forall x \in B_{r_y}(y)$

cover X with $\bigcup_{y \in X} B_{r_y}(y)$, and pick finite subcover. $B_{r_{y_1}}(y_1) \cup \cdots B_{r_{y_N}}(y_N)$

define $g_z(x) = \max_{j=1,\dots N} g_{y_j z}(x) \in \bar{\mathcal{A}}$ by Lemma 4.

$g_z(z) = f(z) = g_{y_j z}(z) \ \forall j$

$g_z(x) \geq g_{y_j z}(x) \geq f(x) - \varepsilon$ on $B_{r_{y_j}}(y_j)$, and these balls cover X ∎

<u>Step 2</u> $\forall z \in X \ \exists \ r_z$ s.t. $g_z(x) \leq f(x) + \varepsilon \quad \forall x \in B_{r_z}(z)$

Cover X with $\bigcup_{z \in X} B_{r_z}(z)$, let

$\bigcup_{j=1}^{M} B_{r_{z_j}}(z_j)$ be finite subcover

define $g(x) = \min_{j=1..M} g_{z_j}(x) \in \bar{\mathcal{A}}$ by Lemma 4

$g(x) \geq f(x) - \varepsilon \quad \forall x \in X$ since each $g_{z_j}(x) \geq f(x) - \varepsilon$.

$g(x) \leq g_{z_j}(x) \leq f(x) + \varepsilon$ on $B_{r_j}(z_j)$, and these balls cover X

$\Rightarrow \|f - g\|_{sup} \leq \varepsilon$

∎