Math 5210
Monday March 30
Measure theory (Royden Chptr 3), con't.

Recall, we're trying to "measure" something like "length" for arbitrary sets in $\mathbb{R}$ with the outer measure

$$m^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) \text{ s.t. } A \subset \bigcup_{n=1}^{\infty} I_n, \text{ and each } I_n = (a_n, b_n) \text{ is an open interval with } \ell(I_n) = b_n - a_n \right\}$$

We've shown to the dotted line:

prop 1 A countable $\Rightarrow m^*(A) = 0$

prop 2 $A \subset B \Rightarrow m^*(A) \leq m^*(B)$

prop 4a (numbering to correspond to Friday notes)
If I is a finite interval, then $m^*(I) = \ell(I)$

prop 4b If I is an infinite interval then $m^*(I) = \infty = \ell(I)$
check this:

prop 3 $m^*(A \cup B) \leq m^*(A) + m^*(B)$ ← "subadditivity"
more generally,

$$m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m^*(A_k)$$

proof: Let $A, B \subset \mathbb{R}$.
if $m^*(A)$ or $m^*(B) = \infty$, then $m^*(A \cup B) \leq \infty$, so est holds.
Otherwise, Let $\varepsilon > 0$.
Let $A \subset \bigcup_{j=1}^{\infty} I_{1,j}$, $\quad \sum_j \ell(I_{1,j}) \leq m^*(A) + \varepsilon/2$
$B \subset \bigcup_{j=1}^{\infty} I_{2,j}$, $\quad \sum_j \ell(I_{2,j}) \leq m^*(B) + \varepsilon/2$
Then $A \cup B \subset \bigcup_j (I_{1,j} \cup I_{2,j})$, so $m^*(A \cup B) \leq \sum_j \ell(I_{1,j}) + \ell(I_{2,j})$
$\leq m^*(A) + m^*(B) + \varepsilon$ ∎

Turn this into a proof for countable subadditivity:

Example the irrationals in $[0,1]$, $\overset{A}{\overset{\shortparallel}{\mathbb{I}r \cap [0,1]}}$ has $m^*(A) = 1$

proof $m^*(A) \leq 1$ since $A \subset [0,1]$
but $m^*([0,1]) \leq m^*(A) + \underset{\overset{\shortparallel}{0}}{m^*(\mathbb{Q} \cap [0,1])}$, so $1 \leq m^*(A)$ ∎

prop 5 Let $A+y = \{x = a+y \text{ s.t. } a \in A\}$
Then $m^*(A+y) = m^*(A)$, is in your HW set.

We would like that A, B disjoint $\Rightarrow m^*(A \cup B) = m^*(A) + m^*(B)$
Actually we'd like more!: $\{A_k\}$ disjoint $\Rightarrow m^*(\cup A_k) = \sum_k m^*(A_k)$
(This is called countable additivity)
Alas, finite and ctble additivity won't hold for all sets... however:

Theorem Let $A, B \subset \mathbb{R}$ be separated by a positive distance $\delta > 0$,
i.e. $|b-a| \geq \delta \quad \forall a \in A, b \in B$.

Then $m^*(A \cup B) = m^*(A) + m^*(B)$

Lemma for any $A \subset \mathbb{R}$, $m^*(A) = \inf \left\{ \sum_n \ell(I_n) \text{ s.t. } I_n = [a_n, b_n] \text{ is closed and } A \subset \bigcup I_n \right\}$.

proof: call the outer measure
using closed sets $m^{**}(A)$.
then for any open cover of A with intervals
the corresponding ~~co~~ closed intervals also cover A,
so $m^{**}(A) \leq m^*(A)$.

Conversely, if $A \subset \bigcup_n I_n$, closed intervals $I_n = [a_n, b_n]$
Then $\forall \varepsilon > 0$, $A \subset \bigcup_n (a_n - \varepsilon/2^n, b_n + \varepsilon/2^n)$

so $m^{**}(A) \leq m^*(A) + \varepsilon$ ∎

Proof of Thm:

Let A, B separated by $\delta > 0$

Know $m^*(A \cup B) \leq m^*(A) + m^*(B)$

Need reverse. May assume $m^*(A \cup B) < \infty$.

Let $\varepsilon > 0$. Let $A \cup B \subset \bigcup I_n$,

$$\sum \ell(I_n) \leq m^*(A \cup B) + \varepsilon.$$

We may assume each I_n is closed, and that we have "partitioned" longer intervals (and renumbered), so that $\ell(I_n) < \delta \ \forall n$. This does not change $\sum_n \ell(I_n)$.

If $\ell(I_n) < \delta \ \forall n$ then each I_n intersects at most one of A or B. Thus we may extract a cover for A and a cover for B from $\{I_n\}$ without using any interval more than once.

$$\Rightarrow m^*(A) + m^*(B) \leq \sum_{I_{n_j} \cap A \neq \emptyset} \ell(I_{n_j}) + \sum_{I_{n_k} \cap B \neq \emptyset} \ell(I_{n_k}) \leq \sum_n \ell(I_n) \leq m^*(A \cup B) + \varepsilon$$

▥

Important Def: $E \subset \mathbb{R}$ is measurable iff

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \qquad \forall A \subset \mathbb{R}$$

Remarks

- You should think of this condition as saying that we can use E to subdivide sets A and deduce the outer measure of A from the outer measure of the smaller pieces.
- It will turn out that lots of sets are measurable, starting with intervals. The collection of measurable sets turns out to be closed under Boolean operations (union, intersection, complement), as well as under countable unions, i.e.

$$\{E_k\} \text{ measurable} \Rightarrow \bigcup_k E_k \text{ measurable.}$$

  This makes the measurable sets a σ-algebra (any collection closed under $\cup, \cap, (\ )^c$, and ctble union)

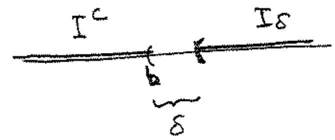
  All open and closed sets will be measurable, for example.

- Measurable sets will have the countable additivity property, and when we restrict m^* to these sets we will call it m, Lebesgue measure.

Results for measurable sets

prop 1 E measurable iff E^c measurable (Royden uses $\tilde{E}$ for E^c)

prop 2 Let $I = [b, \infty)$ or (b, ∞).
Then I is measurable.



Proof: Let $I_\delta = [b+\delta, \infty)$ (for any $\delta > 0$).
Then I^c and I_δ are separated by $\delta/2$ so $A \cap I_\delta$ and $A \cap I^c$ also are.

Thus
$$m^*(A) \geq m^*((A\cap I_\delta) \cup (A \cap I^c)) = \underbrace{m^*(A\cap I_\delta)}_{\geq\, m^*(A\cap I) - \delta} + m^*(A\cap I^c)$$
by separation theorem

$m^*(A\cap I) - \delta$, because $m^*(A\cap I) \leq m^*(A \cap [b, b+\delta]) + m^*(A \cap I^\delta) \leq \delta + m^*(A\cap I^\delta)$

Thus $m^*(A) \geq m^*(A\cap I) + m^*(A \cap I^c)$.
$m^*(A) \leq m^*(A \cap I) + m^*(A\cap I^c)$ is just subadditivity. ■

Cor all half-infinite intervals are measurable.

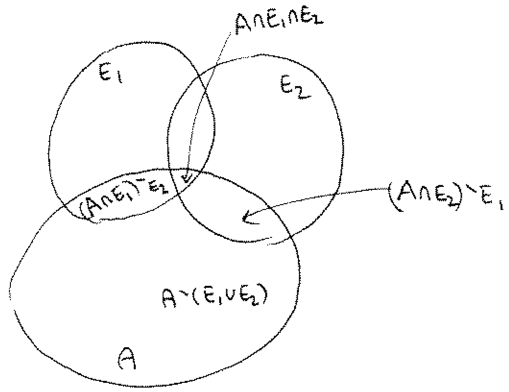
On the next page we prove that the collection of measurable sets is closed under pairwise union and intersection. As a Corollary, we get

prop 3 every interval is measurable
proof:

<u>Def</u> $\mathcal{M} := \{\text{measurable subsets of } \mathbb{R}\}$

<u>Theorem</u> Let $E_1, E_2 \in \mathcal{M}$, $A \subset \mathbb{R}$

(1)



then you can measure A by subdividing it into 4 pieces, using E_1 and E_2:

(1a) $m^*(A) = m^*(A \setminus (E_1 \cup E_2)) + m^*((A \cap E_1) \setminus E_2)$
$+ m^*(A \cap E_1 \cap E_2)$
$+ m^*((A \cap E_2) \setminus E_1)$

as a consequence,

(1b) $m^*(A) = m^*(A \setminus (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2))$

(2) Thus $\left.\begin{array}{l} E_1 \cup E_2 \in \mathcal{M} \\ E_1 \cap E_2 \in \mathcal{M} \\ E_1^c \in \mathcal{M} \end{array}\right\}$ so $\boxed{\mathcal{M} \text{ is a Boolean algebra}}$

(closed wrt $\cap, \cup, {}^c$)

<u>proof:</u> this is fun! ☺