Math 5210
Tuesday ~~Feb.~~ March 3

Convolution and approximation theorems, §7.5

Hint on 7.4.5 #4:
$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots = f(x)$

1st show $R = 1$ for the power series $f(x)$

2nd By differentiating term by term, show $f(x)$ satisfies the same 1st order IVP as $g(x) = (1+x)^\alpha$, i.e.

$$\begin{cases} y' = \frac{\alpha y}{1+x} \\ y(0) = 1. \end{cases}$$

Use IVP ! thm to deduce $f = g$ on the interval $|x| < 1$!!

Let $f, g: \mathbb{R} \to \mathbb{R}$ continuous, with at least one of f or $g \equiv 0$ outside of a bounded set.
(Actually, one of f or g could be a scalar fcn, with range space $\mathbb{R}$, and the other one could map to a Banach space X.)

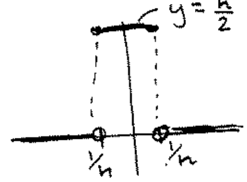
The convolution function $f * g$ is defined by

$$f*g(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$$
$$= \int_{-\infty}^{\infty} g(x-y)\, f(y)\, dy$$
$$= f*g(x)$$

check!, also note convolution is linear in each factor.

(this is not exactly the same convolution of functions you did in 2280, DE's Laplace transform, but it's similar)

Example if $g_n(y)$ was the piecewise continuous function with integral 1, and graph


$y = \frac{n}{2}$
$\frac{1}{n}$ $\frac{1}{n}$

then
$$f*g_n(x) = \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} f(x-y)\, dy$$
$$= \frac{1}{2/n}\int_{-\frac{1}{n}}^{\frac{1}{n}} f(x-y)\, dy.$$

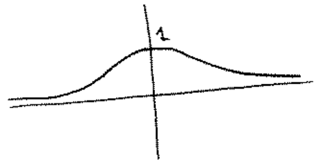
is an average of f, centered at x, and $f*g_n(x) \xrightarrow{n\to\infty} f(x)$, for f continuous.

hence the word "convolution", because $f*g(x)$ sort of averages ~~and~~ samples of $f \cdot g$, except that f & g are evaluated at different points, so the averaging is mixed up

For the application we have in mind (Weierstrass approx. theorem) we will consider a similar family of g's, called an <u>approximate identity</u> family

Start with

$g(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$, the Gaussian:



- $g(x) = g(-x)$
- $g(x) \geq 0$
- $\int_{-\infty}^{\infty} g(x)\,dx = 1$

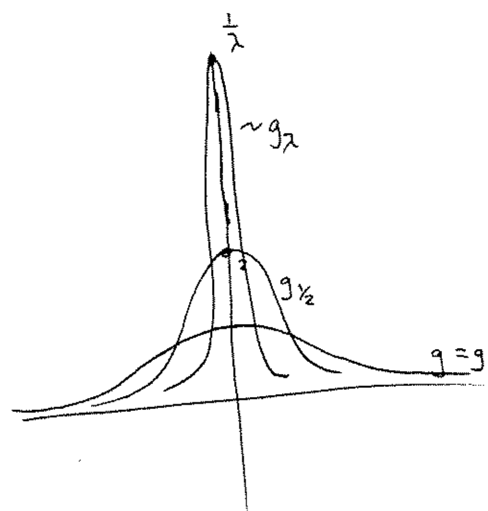
↳ this is a 2210 computation every mathematician should know! it uses polar coords!

proof:

Now scale these with a real parameter $\lambda > 0$, and consider what happens as $\lambda \to 0$:

$g_\lambda(x) := \frac{1}{\lambda} g\left(\frac{x}{\lambda}\right)$; $g_\lambda \geq 0$, $g_\lambda(-x) = g_\lambda(x)$, and:

- $\int_{-\infty}^{\infty} g_\lambda(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\lambda} g\left(\frac{x}{\lambda}\right) dx \underset{\substack{u = \frac{x}{\lambda} \\ du = \frac{1}{\lambda}dx}}{=} \int_{-\infty}^{\infty} g(u)\,du = 1$



As $\lambda \to 0^+$ we expect $(f * g_\lambda)(x) \to f(x)$ for cont. f, since we are computing weighted averages of f, sampling closer & closer to $f(x)$.

__Theorem 1.__ Let $f:\mathbb{R}\to X_{\text{Banach}}$, f cont, $f\equiv 0$ outside $[-M,M]$ some $M\in\mathbb{R}$

(the $\overline{\{x \mid f(x)\neq 0\}}$ is called the __support__ of f, we are considering f with __compact support__)

Let g and g_λ as on previous page.

Then $\{f * g_\lambda\} \to f$ as $\|\cdot\|_{sup}$ as $\lambda \searrow 0^+$.

__proof__:

$$f * g_\lambda(x) - f(x) = \int_{-\infty}^{\infty} (f(x-y) - f(x))\, g_\lambda(y)\, dy \qquad \text{why?}$$

Note: Since f is continuous with compact support, f is uniformly continuous, and bounded.

Let $\varepsilon > 0$. Pick δ s.t. $|z-w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon/2$.

$$f * g_\lambda(x) - f(x) = \underbrace{\int_{|y|<\delta} (f(x-y) - f(x))\, g_\lambda(y)\, dy}_{I_1} + \underbrace{\int_{|y|\geq\delta} (f(x-y) - f(x))\, g_\lambda(y)\, dy}_{I_2}$$

$$|I_1| \leq \int_{|y|<\delta} \varepsilon/2\; g_\lambda(y)\, dy < \frac{\varepsilon}{2}\int_{\mathbb{R}} g_\lambda(y)\, dy = \varepsilon/2$$

To estimate I_2, let C be a bound for f, $|f(z)| \leq C \;\; \forall z \in \mathbb{R}$.

$$|I_2| \leq \int_{|y|\geq\delta} 2C\, g_\lambda(y)\, dy = 2C \int_{|y|\geq\delta} \frac{1}{\lambda} g\!\left(\frac{y}{\lambda}\right) dy = \int_{\substack{|x\lambda|\geq\delta \\ \text{i.e.} \\ |x|\geq \frac{\delta}{\lambda}}} \overbrace{g(x)}^{\frac{1}{\sqrt{\pi}}e^{-x^2}}\, dx \longrightarrow 0 \text{ as } \lambda \searrow 0^+$$

$x = \frac{y}{\lambda}$

$dx = \frac{1}{\lambda} dy$

So pick λ_0 s.t. $0<\lambda<\lambda_0 \Rightarrow |I_2| < \varepsilon/2$

Thus $0<\lambda<\lambda_0 \Rightarrow$ ~~Thus~~ $|f * g_\lambda(x) - f(x)| \leq |I_1| + |I_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon$!

$$\Rightarrow \|f * g_\lambda(x) - f(x)\|_{sup} \leq \varepsilon$$

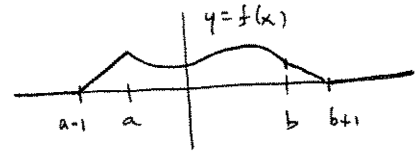
∎

Theorem 2 (Weirestrass Approximation Theorem)

Let $f:[a,b]\to\mathbb{R}$ continuous
Let $\varepsilon>0$. Then $\exists$ a polynomial $p_n(x)=\sum_{k=0}^{n} a_k x^k$ s.t. $\|f-p_n\|_{sup}<\varepsilon$ on the interval $[a,b]$.

Note: f is only continuous, so there's no hope trying Taylor series! (and these don't even always work even for C^∞ functions)

proof: • Extend f to be a continuous function on $\mathbb{R}$, identically zero for $|x|\geq M$, some $M<\infty$. For example, you could use piecewise-linear extensions as indicated



• Pick a small $\lambda>0$ s.t.
$$\|f*g_\lambda - f\|_{\substack{sup\\(\text{over } \mathbb{R})}} < \varepsilon/2 \qquad \text{by Theorem 1.}$$

• Punchline! Approximate $g_\lambda(z)$ with it's Taylor series:
$$g_\lambda(z)=\frac{1}{\sqrt{\pi}}\frac{1}{\lambda}e^{-(z/\lambda)^2}=\sum_{k=0}^{\infty}a_k z^k \qquad \left(=\frac{1}{\sqrt{\pi}}\frac{1}{\lambda}\left(1+\left(-\left(\tfrac{z}{\lambda}\right)^2\right)+\frac{1}{2!}\left(-\left(\tfrac{z}{\lambda}\right)^2\right)^2+\dots\right)\right)$$
$$=\lim_{n\to\infty}\sum_{k=0}^{n}a_k z^k=\lim_{n\to\infty}S_n(z) \qquad \text{i.e. } e^x \text{ with } x=-(z/\lambda)^2$$

converges uniformly for $|z|\leq 2M$ since e^x power series has $R=\infty$, so this one does too.

Then for $|x|\leq M$,
$$f*g_\lambda(x)=\underbrace{f*(S_n(x))}_{\substack{\| \\ \int_{-M}^{M} f(y)\left(\sum_{k=0}^{n}a_k(x-y)^k\right)dy}} + \underbrace{f*(g_\lambda-S_n)(x)}_{\text{error}}$$

$$\underbrace{\int_{-M}^{M} f(y)\left(\sum_{k=0}^{n}a_k(x-y)^k\right)dy}_{:=p_n(x),\text{ is a degree }\leq n\text{ polynomial!}}$$

error: $|\text{Error}|\leq \int_{-M}^{M}\underbrace{|f(y)|}_{\leq C}\underbrace{|(g_\lambda-S_n)(x-y)|}_{\substack{|x-y|\leq 2M\\ \text{so this is}\\ \leq \frac{\varepsilon}{4CM}\\ \text{for } n\geq n_0,\\ \text{some } n_0}}dy$

$\Rightarrow |\text{Error}|\leq \varepsilon/2$

Finally, for $|x|\leq M$,
$$|f(x)-p_n(x)|=|f(x)-f*g_\lambda(x)+f*g_\lambda(x)-p_n(x)|$$
$$<\varepsilon/2+\varepsilon/2=\varepsilon$$