Math 5210
Friday March 27

HW for Friday April 3
Royden p.56 6, 7, 8
p.62 10, 11, 13, 14b
Strichartz p.654 2, 3

Measure theory – chapter 3 Royden

Introduction

We are now starting measure theory and the Lebesgue integral.
You might want to read the motivation in §14.1 of the text, starting p 623.
Also, the Royden xeroxes, which I plan to follow. We will also go through §14.2 of Strichartz: general outer, and metric outer measures, with Hausdorff measures as an example.

The Riemann integral is great for

$$f : \mathbb{R} \to X_{\text{Banach}}$$

as long as f is continuous (or even piecewise continuous)

It turns out the Lebesgue integral is great for

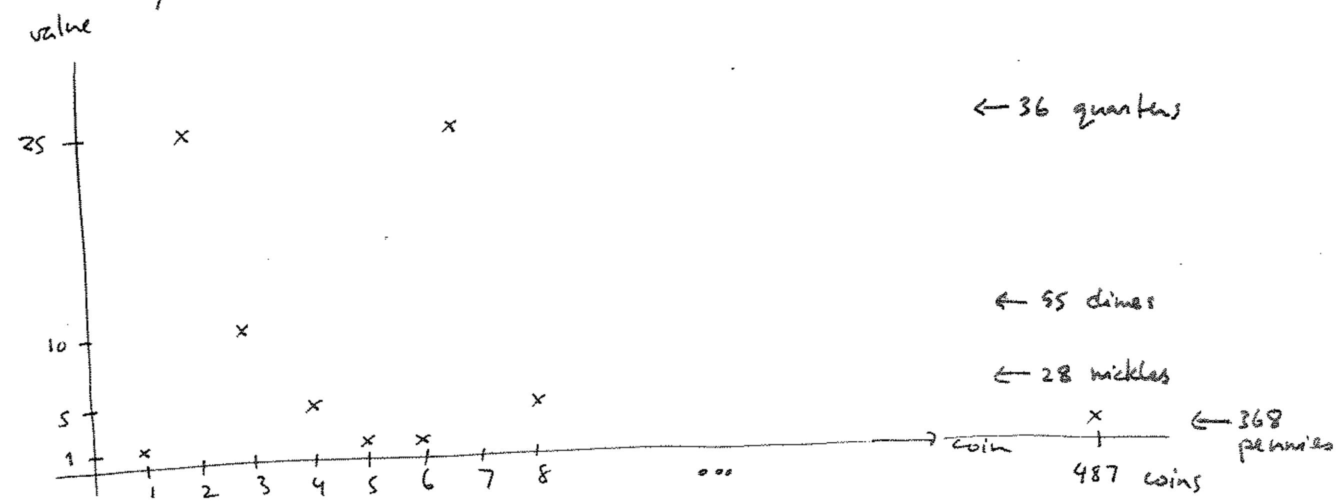
$$f : X \to \mathbb{R}, \qquad \text{as long as } f \text{ is "measurable."}$$

X ↑ "measure space"

For $f : \mathbb{R} \to \mathbb{R}$ (or $\mathbb{R}^n \to \mathbb{R}$), measurable is a much less restrictive condition than continuous, so the Lebesgue integral is much better.

But you can see the difference between the Lebesgue & Riemann integrals intuitively in the following example. It illustrates that in Riemann integration you partition the domain, whereas in Lebesgue integration you partition the range .... for "nice" fcns you get the same answer either way.

Example 1
You have a bowl full of change.
How would you count its value?



Riemann: $1 \cdot 1 + 1 \cdot 25 + 1 \cdot 10 + 1 \cdot 5 + 1 \cdot 1 + 1 \cdot 1 + \cdots$

Lebesgue: $25(36) + 10(55) + 5(28) + 1(368)$

Example 2 : Let $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Is there a sensible way to define

$$\int_0^1 f(x)dx ?$$

- Riemann integral DNE since $\exists$ arbitrarily "fine" partitions for which the Riemann sums can take on any value between 0 & 1.

- But there are many more irrats than rats, since the rats are countable but the irrats are not.

So perhaps $\int_0^1 f(x)dx = 0$ would be correct?

We need a way to "measure" the size of subsets of $\mathbb{R}$, that generalizes length for intervals.

$$\boxed{\text{Def: Let } A \subset \mathbb{R} \quad \text{the outer measure} \quad m^*(A) := \inf_{A \subset \bigcup I_n} \left(\sum_{n=1}^{\infty} \ell(I_n) \right)}$$

$\ell(I_n) := b_n - a_n$ if I_n ~~has en~~ $= (a_n, b_n)$

open intervals, countable (including finite) union

use the word "outer" because we are getting the measure by covering A with (open) ~~sets inter~~ sets, i.e. measuring from the outside. If we tried to measure A ~~using~~ from inside (using closed subsets) ~~[illegible]~~ we would call it an "inner measure".

example : $m^*(\mathbb{Q}) = 0$

pf: Let $\mathbb{Q} = \{q_1, q_2, \dots\}$

Let $\varepsilon > 0$.

Let $I_n = (q_n - \frac{\varepsilon}{2^n}, q_n + \frac{\varepsilon}{2^n}) \subset \mathbb{R}$; $\ell(I_n) = \frac{\varepsilon}{2^{n-1}}$

Then $\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_n$, $\sum_{n=1}^{\infty} \ell(I_n) = \varepsilon(~~[illegible]~~ 2 + 1 + \frac{1}{2} + \dots) = 4\varepsilon$.

Since ε is arbitrary, $m^*(\mathbb{Q}) = 0$.

Of course this would work for any countable set.

quick properties from the definition of outer measure :

(1) $m^*(A\cup B) \le m^*(A) + m^*(B)$

(2) $A \subset B \Rightarrow m^*(A) \le m^*(B)$

<u>example</u> : $m^*([0,1]) = 1$! First note $m^*([0,1]) \le 1$ since $[0,1] \subset (-\varepsilon, 1+\varepsilon)$, $\varepsilon > 0$ and $\ell((-\varepsilon, 1+\varepsilon)) = 1 + 2\varepsilon$.

proof : Let $[0,1] \subset \bigcup_{n=1}^{\infty} I_n$. Suffices to show $\sum \ell(I_n) \ge 1$.

By Heine-Borel, $\exists$ finite subcover, $\bigcup_{n=1}^{N} I_n$.

pick one of these intervals containing 0,

$0 \in (a_1, b_1) = I_1$ (renumber)

If $b_1 > 1$, $\ell(I_1) > 1 \Rightarrow \sum \ell(I_n) > 1$.

else $b_1 \in (a_2, b_2) := I_2$ (renumber)

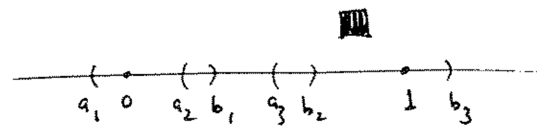
If $b_2 > 1$, $\ell(I_1) + \ell(I_2) = (b_1 - a_1) + (b_2 - a_2)$

$$> \underset{\overset{\vee}{0}}{-a_1} + \underset{\overset{\vee}{0}}{(b_1 - a_2)} + \underset{\overset{\vee}{1}}{b_2} > 1 \Rightarrow \sum \ell(I_n) > 1.$$

continue until you stop , with some $I_m = (a_m, b_m)$, $b_m > 1$.

Then
$$\sum_{n=1}^{m} \ell(I_n) = b_1 - a_1 + b_2 - a_2 + \dots + b_m - a_m$$
$$= \underset{\overset{\vee}{0}}{-a_1} + \underset{\overset{\vee}{0}}{(b_1 - a_2)} + \underset{\overset{\vee}{0}}{(b_2 - a_3)} + \dots + \underset{\overset{\vee}{0}}{(b_{m-1} - a_m)} + \underset{\overset{\vee}{1}}{b_m} > 1.$$

this argument would show that any finite closed interval $[a,b]$ has $m^*[a,b] = b - a$



$\boxed{\text{So, returning to example 2,}}$

$m^*(\mathbb{Q}) = 0 \Rightarrow m^*(\mathbb{Q} \cap [0,1]) = 0$

$m^*([0,1]) = 1$

so $m^*(\mathbb{I}r \cap [0,1]) \le m^*([0,1]) = 1$

but $\underset{\overset{\parallel}{1}}{m^*([0,1])} \le m^*(\mathbb{I}r \cap [0,1]) + \overset{0}{\cancel{m^*(\mathbb{Q} \cap [0,1])}}$

$\Rightarrow m^*(\mathbb{I}r \cap [0,1]) = 1$

so, $\int_0^1 f(x)\,dx$ should equal $1 \cdot m^*(\mathbb{Q} \cap [0,1]) + 0 \cdot m^*(\mathbb{I}r \cap [0,1])$

$= 1 \cdot 0 + 0 \cdot 1$

$= 0.$

And for the Lebesgue integral, it will!

Collecting what we've shown:

for the outer measure m^* defined for all subsets of $\mathbb{R}$ on page 2, we know:

<u>prop1</u> A countable $\Rightarrow m^*(A) = 0$

<u>prop2</u> $A \subset B \Rightarrow m^*(A) \leq m^*(B)$

<u>prop3</u> $m^*(A \cup B) \leq m^*(A) + m^*(B)$

in fact, this countable sub-additivity extends:

$$\boxed{m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m^*(A_k)}$$

<u>proof</u>: If any $m^*(A_k) = \infty$, then both sides of this ineq $= \infty$, so true!

Otherwise let $\varepsilon > 0$. Pick $\{I_{k,j}\}_{j=1}^{\infty}$ cover A_k, $\sum_j \ell(I_{k,j}) < m^*(A_k) + \varepsilon/2^k$

then $\bigcup_{j,k=1}^{\infty} I_{k,j} \supset \bigcup_k A_k$

and
$$\sum_{j,k} \ell(I_{k,j}) = \sum_k \left(\sum_j \ell(I_{k,j})\right)$$
$$\leq \sum_k m^*(A_k) + \varepsilon/2^k$$
$$= \left(\sum_k m^*(A_k)\right) + \varepsilon$$

so $m\left(\bigcup A_k\right) \leq \left(\sum_k m^*(A_k)\right) + \varepsilon$ ■

<u>prop4</u> If A is an interval, then
$m^*(A) = \ell(A)$.

we did this for a finite interval on page 2.
If A is (at least) half infinite, then it has finite intervals of arbitrarily large length as subsets.
So it has subsets of arbitrarily large outer measure as subsets.
so prop2 $\Rightarrow m^*(A) = \infty$.

$m^*([a,b]) = b-a$

so $\underbrace{m^*((a,b]),\ m^*((a,b)),\ m^*([a,b))}_{} \leq b-a$

$\vee\!/$

$m^*([a+\varepsilon, b-\varepsilon])$

$\shortparallel$

$b-a-2\varepsilon \quad (\varepsilon < \frac{b-a}{2})$

Let $\varepsilon \to 0$. ■

to be cont'd!