Math 5210
Wed. March 25

HW for Friday: cancel 12.2 #5.
We start Chapter 14 (Measure theory) on Friday!

- Prove Theorem 2 Tuesday notes,
  about uniform convergence of Fourier series of $f \in C^1(\mathbb{R})$ & 2π-periodic.
  (and ptwise est. if $f'(x_0)\ \exists$, f only cont.)

- Although $f \in C(\mathbb{R})$ & 2π-periodic does not imply the Fourier series for f converges uniformly (or even pointwise) to f, the text does prove the following interesting result:

Write $\quad S_n(f) := \mathrm{proj}_{V_n} f = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + \sum_{k=1}^{n} b_k \sin kx$

$$= \sum_{k=-n}^{n} c_k e^{ikx}$$

Define $\quad \sigma_N(f) = \frac{1}{N+1} \sum_{n=0}^{N} S_n(f)$

Then $\sigma_N(f) \to f$ uniformly $\quad (f \in C(\mathbb{R}))$. $\quad$ (We'll skip this.)

- Complex inner product spaces:

  We skipped this discussion in Chapter 9, but you may wish to read pages 364-366.
  If V is a complex scalar vector space, then a complex inner product satisfies
  (which has values in $\mathbb{C}$, not $\mathbb{R}$)

  1. $\langle x, y\rangle = \overline{\langle y, x\rangle}$ $\quad$ Hermetian symmetry.
  2. $\langle ax+by, z\rangle = a\langle x,z\rangle + b\langle y,z\rangle$
     $\langle x, ay+bz\rangle = \bar{a}\langle x,y\rangle + \bar{b}\langle x,z\rangle$ $\quad$ Hermetian linearity.
  3. $\langle x,x\rangle$ is real and ≥ 0. $\quad \langle x,x\rangle = 0$ iff $x=0$.

Examples
- $\mathbb{C}^n = \left\{ \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \text{ s.t. } z_i \in \mathbb{C} \right\}$. $\quad \langle v, w\rangle := \sum_{j=1}^{n} v_j \overline{w_j}$

- $C([-\pi,\pi], \mathbb{C})$ $\quad \langle f, g\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, \bar{g}(x)\, dx$

  (for chapter 12)

- If V is a complex scalar vector space, with a complex inner product, then you can still define a norm

$$\|x\| := \langle x, x\rangle^{1/2}$$

still prove C.S. $\quad |\langle x,y\rangle| \leq \|x\| \cdot \|y\|$

so Δ ineq: $\quad \|x+y\| \leq \|x\| + \|y\|$.

Furthermore, define $x \perp y$ iff $\langle x,y\rangle = 0$.

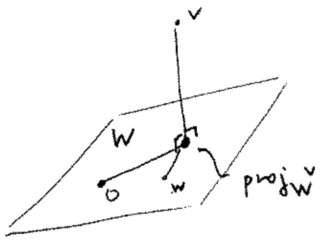
Pythag. thm still holds: $x \perp y \Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$

Therefore projection theorem still holds:

If $W \subset V$ is a subspace with o.n. basis $\{u_1, \dots u_N\}$

then $\text{proj}_W v := \sum_{j=1}^{N} \langle v, u_j\rangle u_j$ is the closest point in W to v

because $z := v - \text{proj}_W v$ is $\perp$ to all $w \in W$



(notice, $\text{proj}_W v \neq \sum \langle u_j, v\rangle u_j$!)

Thus,

$$\|v\|^2 = \underbrace{\|\text{proj}_W v\|^2}_{= \sum_{j=1}^{N} |\langle v,u_j\rangle|^2} + \|v - \text{proj}_W v\|^2$$

- Fourier series look cleaner using the complex inner product (& allowing $\mathbb{C}$-valued fns)

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\bar{g}(x)\,dx$$

(vs. our $\mathbb{R}$-inner product $\langle f,g\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$)

Since now $\{e^{ikx}\}_{k\in\mathbb{Z}}$ are orthonormal

& $c_k := \langle f, e^{ikx}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx$

this is the deeper reason why (for a real or complex-valued fcn f),

$$S_n(f) = \sum_{k=-n}^{n} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + \sum_{k=1}^{n} b_k \sin kx$$

because your just using different orthogonal bases to project onto the same complex subspace. (Of course, it's also just a simple computation with Euler's formula to check this identity.)

Also, Parceval's equality (conseq. of $S_n(f) \xrightarrow{L_2} f$) looks better in complex space:

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \sum_{-\infty}^{\infty} |c_k|^2}$$