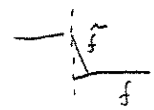
Math 5210
Tuesday March 24
12.2 - 12.3

- Finish Monday's notes, that for 2π-periodic continuous f,

$$\boxed{\operatorname{proj}_{V_n} f = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + \sum_{k=1}^{n} b_k \sin kx \xrightarrow[L^2]{} f}$$

<u>Remarks</u> • If f is $\underbrace{\text{piecewise cont}}_{\substack{\text{jump discontinuities}\\ \text{at a finite \# of points}\\ \text{between } -\pi \text{ \& } \pi.}}$ & 2π-periodic, the same result holds



Because $\forall \varepsilon > 0 \ \exists \ \tilde{f}$, $\|f - \tilde{f}\|_{L^2} < \varepsilon$
$\tilde{f}$ cont & 2π-periodic.

$\Rightarrow \exists \ P_N(x) =$ trig poly of "order" N,
$\|P_N - \tilde{f}\|_{L^2} < \varepsilon$

$\Rightarrow \|P_N - f\|_{L^2} < 2\varepsilon \Rightarrow \forall n \geq N, \ \|\operatorname{proj}_{V_n} f - f\|_{L^2} < 2\varepsilon.$

- $\operatorname{proj}_{V_n} f \xrightarrow[L^2]{} f \Rightarrow \underbrace{\|\operatorname{proj}_{V_n} f\|_{L^2}^2}_{} \longrightarrow \|f\|_{L^2}^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx$

$= \frac{a_0^2}{2} + \sum_{k=1}^{n} a_k^2 + b_k^2$. <u>Cor</u> $\|f\|_{L^2}^2 = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 + b_k^2$

$\Rightarrow \{a_k\}, \{b_k\} \to 0$ as $k \to \infty$ (<u>Riemann-Lebesgue Lemma</u>)

- Complex form:

for $k \in \mathbb{Z}$, $c_k := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-ikx}\,dx$ $\quad \left(= \frac{1}{2}(a_k - i b_k)\right)$ $k \neq 0$
$\left(\frac{a_0}{2}, k = 0\right)$

then $\boxed{\operatorname{proj}_{V_n} f = \sum_{k=-n}^{n} c_k e^{ikx}}$ $\longleftarrow$ (you check this in HW) use Euler

What about sup norm or pointwise convergence?

<u>Theorem 1</u> (easy) Let $f \in \underset{\substack{\uparrow\\ \text{twice continuously}\\ \text{differentiable}}}{C^2}(\mathbb{R})$, and 2π-periodic. Then $\operatorname{proj}_{V_n} f \to f$ <u>uniformly</u>

<u>proof</u> $2\pi c_k = \int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx$; for $k \neq 0$ integrate by parts twice

$$= \underbrace{\cancel{f \frac{e^{-ikx}}{-ik}\Big]_{-\pi}^{\pi}}}_{0} + \int_{-\pi}^{\pi} f' \frac{e^{-ikx}}{ik}\,dx = \underbrace{\cancel{f' \frac{e^{-ikx}}{k^2}\Big]_{-\pi}^{\pi}}}_{0} - \int_{-\pi}^{\pi} \frac{f'' e^{-ikx}}{k^2}\,dx$$

Thus $|c_k| \le \frac{M}{k^2}$ $\qquad \left(M = \max_{x \in [-\pi,\pi]} |f''(x)| \right)$

$\Rightarrow \operatorname{proj}_{V_n} f \xrightarrow{\text{uniformly absolutely}}$ to a limit function (continuous)

Since $\operatorname{proj}_{V_n} f \xrightarrow{L^2} f$, this limit function must equal f! ∎

<u>Theorem 2</u> (Much harder than Theorem 1).

Let $f \in C'(\mathbb{R})$ be 2π-periodic.

Then $\operatorname{proj}_{V_n} f \to f$ uniformly

(and, if $f \in C(\mathbb{R})$ is 2π-periodic, and $f'(x_0)$ exists, then $(\operatorname{proj}_{V_n} f)(x_0) \to f(x_0)$ pointwise)

I think I assigned this as HW! Today's notes should help

<u>proof</u>: (uses convolution ideas)

$$(\operatorname{proj}_{V_n} f)(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

$$= \sum_{k=-n}^{n} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky}\, dy \right] e^{ikx}$$

$$= \sum_{k=-n}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{ik(x-y)}\, dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \underbrace{\left(\sum_{k=-n}^{n} e^{ik(x-y)} \right)}_{:D_n(x-y)} dy,$$

for the <u>Dirichlet kernel</u> $\boxed{D_n(t) = \sum_{k=-n}^{n} e^{ikt}}$

$$= \frac{1}{2\pi} f * D_n(x)$$

geometric series!

$$e^{it} D_n(t) = D_n(t) + e^{i(n+1)t} - e^{-int}$$

$$D_n(t) = \frac{e^{i(n+1)t} - e^{-int}}{e^{it} - 1}$$

$$= \frac{e^{it/2}}{e^{it/2}} \left[\frac{e^{i(n+\frac{1}{2})t} - e^{-i(n+\frac{1}{2})t}}{e^{it/2} - e^{-it/2}} \right]$$

$$= \frac{2i \sin(n+\frac{1}{2})t}{2i \sin \frac{1}{2}t}$$

$$\boxed{D_n(t) = \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{1}{2}t)}}$$

Notice, if f, g are 2π-periodic than $f * g = g * f$ (as before, on $\mathbb{R}$)

$$(f*g)(x) = \int_{-\pi}^{\pi} f(y) g(x-y)\, dy$$

$z = x-y$; $y = x-z$

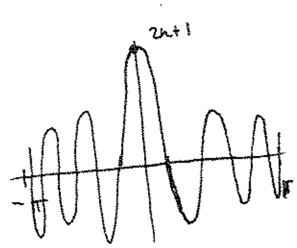
$dz = -dy$

$$= \int_{x+\pi}^{x-\pi} g(z) f(x-z)(-dz)$$

$$= \int_{x-\pi}^{x+\pi} g(z) f(x-z)\, dz = \int_{-\pi}^{\pi} g(z) f(x-z) dz = g * f\ (z)$$

you can integrate a 2π per. fn across any period & get same ans.

graph of $D_n(t)$, large n:



not exactly an approx-id., although

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \underset{\uparrow \, \sum_{k=-n}^{n} e^{ikt}}{D_n(t)}\,dt = 1$$ does hold (only e^{i0t} term contributes)

Thus (writing $S_n(f)$ for $\mathrm{proj}_{V_n} f$),

$$S_n f(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(y) - f(x)]\,[D_n(x-y)]\,dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x-y) - f(x)]\,[\underset{\uparrow \, \frac{\sin(n+\frac{1}{2})y}{\sin\frac{1}{2}y}}{D_n(y)}]\,dy \qquad \text{(using convolution commutes)}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\frac{f(x-y)-f(x)}{\sin\frac{1}{2}y}\right]\left[\sin(n+\tfrac{1}{2})y\right]dy$$

$$= \underset{\substack{\uparrow \\ |y|<\delta}}{I_1} + \underset{\substack{\uparrow \\ |y|>\delta}}{I_2} \quad ; \ \delta \text{ to be determined.}$$

$$I_1 : \ \frac{1}{2\pi}\int_{|y|<\delta} \underbrace{\left[\frac{f(x-y)-f(x)}{-y}\right]}_{\text{a difference quotient for } f'(x)} \underbrace{\left[\frac{-y}{\sin\frac{1}{2}y}\right]}_{\text{uniformly bd for } |y|\le\pi, \text{ by } \pi!} \underbrace{\left[\sin(n+\tfrac{1}{2})y\right]}_{\text{uniformly bd. (by 1).}} dy$$

$$\boxed{\begin{array}{l} \underline{\text{Deduce}}: \text{If } f'(x) \text{ exists, then for } \delta \text{ sufficiently small} \\ \qquad |I_1| \le \frac{2\delta}{2\pi}\,(|f'(x)|+1)(\pi)(1) \\ \text{If } f \in C^1(\mathbb{R}), \\ \qquad |I_1| \le C\delta, \ C \ \underline{\text{independent}} \text{ of } x. \end{array}}$$

← for pointwise result

← for uniform convergence result

$$I_2 = I_2^- + I_2^+$$

$-\pi \le y \le -\delta \qquad \delta \le y \le \pi$.

Estimate analogously in I_2^- and I_2^+

$$I_2^+ : \quad \int_\delta^\pi \underbrace{\left[\frac{f(x-y)-f(x)}{\sin\frac{1}{2}y}\right]}_{\text{uniformly continuous}} \underbrace{\left[\sin(n+\tfrac{1}{2})y\right]}_{\text{oscillates rapidly, with period } \frac{2\pi}{n+\frac{1}{2}}} dy$$

causes cancellation over half periods.

call this term $g(y)$

(Remark for fixed x, in HW, you could just apply Riemann-Lebesgue lemma to deduce that $I_2^+ \to 0$ as $n \to \infty$, since $\sin(n+\frac{1}{2})y = \sin ny \cos\frac{1}{2}y + \cos ny \sin\frac{1}{2}y$ and you can interpret this integral as computing Fourier coef's for a p.w. cont fcn.)

In case $f \in C^1(\mathbb{R})$ we integrate by parts:

$$I_2^+ = g(y)\left[\frac{-\cos(n+\frac{1}{2})y}{n+\frac{1}{2}}\right]_\delta^\pi + \int_\delta^\pi g'(y)\left(\frac{\cos(n+\frac{1}{2})y}{n+\frac{1}{2}}\right) dy$$

$$= \underbrace{g(\delta)\left[\frac{+\cos(n+\frac{1}{2})\delta}{n+\frac{1}{2}}\right]}_{|\ | \le \frac{\pi(|f'(x)|+1)}{n+\frac{1}{2}}}$$

for δ small

$$g'(y) = -\frac{f'(x-y)}{\sin\frac{1}{2}y} + \frac{f(x-y)-f(x)}{(\sin\frac{1}{2}y)^2}(-1)\tfrac{1}{2}\cos\tfrac{1}{2}y$$

If $|f'| \le M$, then

$$|g'| \le \frac{M}{\sin\frac{1}{2}\delta} + \frac{M\pi\frac{1}{2}}{\sin\frac{1}{2}\delta}$$

$$\le \frac{C}{\delta}$$

so 2^{nd} integral, $|\ | \le \dfrac{\tilde{C}}{\delta(n+\frac{1}{2})}$

$\therefore$ $f \in C^1(\mathbb{R})$

$\Rightarrow |I_2^+| \le \dfrac{C_1}{n+\frac{1}{2}}\left(1+\dfrac{1}{\delta}\right)$; also $|I_2^-|$.

pick n large, and $\delta = \frac{1}{\sqrt{n}}$

$\Rightarrow |I_1| + |I_2| \le \dfrac{C_3}{\sqrt{n}} \to 0$ (unif. in x) as $n \to \infty$!!