Math 5210
Monday March 2 §7.4

On Friday we proved ~~two~~ theorems about interchanging limits with integrals and derivatives:

Theorem 1 Let $f_k : [a,b] \to X_{\text{Banach}}$, f_k continuous, $\{f_k\} \to f$ in $\|\cdot\|_{sup}$

Then $\displaystyle\lim_{k\to\infty} \int_a^b f_k(x)dx = \int_a^b f(x)dx$

Theorem 2 Let $\left.\begin{array}{l} f_k : [a,b] \to X \\ f_k' : [a,b] \to X \end{array}\right\}$ both f_k & f_k' continuous.

If $\{f_k\} \to f$, and $\{f_k'\} \to g$ in $\|\cdot\|_{sup}$

Then f' exists, and $f'(t) = g(t) = \lim_{k\to\infty} f_k'(t)$ (is cont.)

On Friday we applied Theorem 1 to DE's.

Today we'll use it to justify termwise differentiation & integration of power series

Def: $\displaystyle f(x) := \sum_{n=0}^{\infty} a_n (x-x_0)^n$ is called a power series, centered at $x_0 \in \mathbb{R}$.

Theorem 3 $\exists!\ R \in [0,\infty]$, called the radius of convergence of f, such that

(a) $|x-x_0| < R \Rightarrow$ series converges absolutely, and this convergence (and absolute convergence) is uniform $\forall\ |x-x_0| \le \mu R$, $0 \le \mu < 1$ fixed.

(b) $|x-x_0| > R \Rightarrow$ series diverges.

proof: Let the series converge at $y \in \mathbb{R}$, i.e. $\sum a_n (y-x_0)^n$ converges.

Thus $\displaystyle\lim_{n\to\infty} |a_n (y-x_0)|^n = 0$.

So $\exists$ M s.t. $|a_n||y-x_0|^n \le M$ $\forall n$

Then if $|x-x_0| = \mu|y-y_0|$ with $0 \le \mu < 1$

we have
$$\sum_{n=0}^{\infty} |a_n||x-x_0|^n = \sum_{n=0}^{\infty} \underbrace{|a_n||y-y_0|^n}_{\le M} \mu^n \le \frac{M}{1-\mu}.$$

$\displaystyle\Rightarrow \sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely. ($\Rightarrow$ conv)

Notice this absolute (and regular) convergence is uniform $\forall\ |x-x_0| \le \mu|y-y_0|$, once $0 \le \mu < 1$ is fixed!

Summary: conv @ $y \Rightarrow$ unif. abs conv $\forall x,\ |x-x_0| \le \mu|y-x_0|$

Contrapositive: If series diverges at x, then it diverges for all y for which $|y-x_0| > |x-x_0|$

Define $R := \sup\{ |x-x_0| \text{ s.t. } \sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges}\}$ $\quad 0 \le R \le \infty$.

$|y-x_0| > R \Rightarrow \sum_{n=0}^{\infty} a_n(y-x_0)^n$ diverges, else R would not be the claimed sup.

$|y-x_0| < R \Rightarrow \exists\, x$ with $|y-x_0| < |x-x_0| \le R$ s.t. $\sum a_n(x-x_0)^n$ CONV
$\Rightarrow \sum a_n(y-x_0)^n$ CONV (absolutely).

As well, convergence is ~~absolute~~ ~~uniform and~~ uniformly absolute $\forall\, y$ s.t. $|y-x_0| \le \mu R$ (for fixed $0 \le \mu < 1$) in case $R < \infty$.

If $R = \infty$, convergence is uniformly absolute $\forall\, y$ s.t. $|y-x_0| \le M$ (for fixed $0 \le M < \infty$).

<u>Theorem 4</u> (sometimes this is useful, although you often just use the ratio test to find radius of convergence)

For $\sum_{n=0}^{\infty} a_n(x-x_0)^n = f(x)$, the radius of convergence

$$R = \tilde{R} := \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

proof: note from page 1 discussion that

$$R = \inf\{ r \mid \limsup_{n\to\infty} |a_n| r^n = +\infty \}$$

and $R = \sup\{ r \mid \limsup_{n\to\infty} |a_n| r^n = 0 \}$

for such an r,

$\limsup_{n\to\infty} |a_n|^{1/n} r \ge 1$

$\Rightarrow \limsup_{n\to\infty} |a_n|^{1/n} \ge \frac{1}{r}$ $\quad \Big\} \, r \searrow R$.

so $\limsup_{n\to\infty} |a_n|^{1/n} \ge \frac{1}{R}$

for such an r

$\limsup_{n\to\infty} |a_n|^{1/n} r \le 1$

so $\limsup_{n\to\infty} |a_n|^{1/n} \le \frac{1}{r}$ $\quad \Big\} \, r \nearrow R$.

so $\limsup_{n\to\infty} |a_n|^{1/n} \le \frac{1}{R}$ ∎

<u>Corollary</u> $\sum_{n=0}^{\infty} a_n(x-x_0)^n$, $\sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$, $\sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1}$

all have the same radius of convergence! (because $\{n^{\frac{1}{n}}\} \to 1$ as $n \to \infty$)

$\Rightarrow$ all converge uniformly for $|x-x_0| \le \mu R$, where R is common radius of convergence and $\mu < 1$ is fixed. ∎

Theorem 5 Let $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ have radius of convergence R.

Then $f'(x)$ and $\int_{x_0}^{x} f(t)\,dt$ exist for for all x s.t. $|x-x_0| < R$,

and
$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^n, \qquad \int_{x_0}^{x} f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-x_0)^{n+1}$$

proof for $0 \le \mu < 1$ and $|x-x_0| \le \mu R$ (analogous cond for $R = \infty$),

$f(x)$ is a uniform limit of the partial sums $\sum_{n=0}^{N} a_n (x-x_0)^n = S_N(x)$

so is cont, and by Theorem 1,

$$\int_{x_0}^{x} f(t)\,dt = \lim_{N\to\infty} \int_{x_0}^{x} \sum_{n=1}^{N} a_n (t-x_0)^n\,dt$$
$$= \lim_{N\to\infty} \sum_{n=1}^{N} \underbrace{\int_{x_0}^{x} a_n (t-x_0)^n\,dt}_{\frac{a_n}{n+1}(x-x_0)^{n+1}}$$

Also, for $|x-x_0| \le \mu R$,
$$S_N'(x) = \sum_{n=1}^{N} a_n n (x-x_0)^n \xrightarrow{\text{unif}} \sum_{n=1}^{\infty} n a_n (x-x_0)^n$$

So by Theorem 2,
$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^n$$

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Examples $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x| < 1 \quad$ geometric series

$$\Rightarrow \ln(1-x) = -\int_0^x \frac{1}{1-t}\,dt = -\int_0^x \sum_{n=0}^{\infty} t^n\,dt = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \qquad |x| < 1$$

$$\left(\frac{1}{1-x}\right)^2 = D_x\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} n x^{n-1} \qquad |x| < 1.$$

Can you:
Find the radius of convergence of $\sum_{n=1}^{\infty} n^2 x^2$ about $x = 0$, and find a closed form expression for this fcn.