Math 5210
Tuesday 3/10

Arzela-Ascoli Theorem §7.6, also 9.3

Let X, Y compact metric spaces
Consider $C(X,Y) = \{f : X \to Y \text{ continuous}\}$
with distance $d_{sup}(f,g) := \sup_{x \in X} d(f(x), g(x))$

- We checked before that if X is compact and Y is complete then this sup distance makes $C(X,Y)$ a complete metric space

<u>proof</u>: Let $\{f_k\}$ Cauchy.

$\forall \varepsilon > 0 \ \exists \ m$ s.t. $k, \ell \geq m \Rightarrow d_{sup}(f_k, f_\ell) < \varepsilon$

$$\rotatebox{90}{$\leq$}$$

$d(f_k(x), f_\ell(x))$ for any $x \in X$

$\Rightarrow \{f_k(x)\} \subset Y$ is Cauchy, so has limit $:= f(x)$.

Then, for ε, m as above, $k, \ell \geq m$

$$\Rightarrow d(f(x), f_k(x)) = \lim_{\ell \to \infty} d(f_\ell(x), f_k(x)) \leq \varepsilon \quad \text{for } k \geq m.$$

$$\Rightarrow d_{sup}(f, f_k) \leq \varepsilon \quad \text{for } k \geq m$$

$\blacksquare$

Oops, must also check that f is continuous:

Let $x \in X$, $\varepsilon > 0$.

Pick k s.t. $d_{sup}(f, f_k) < \varepsilon/3$

and for f_k pick δ s.t. $d(x,z) < \delta \Rightarrow d(f_k(x), f_k(z)) < \varepsilon/3$

then for $d(x,z) < \delta$,

$$d(f(x), f(z)) \leq \underbrace{d(f(x), f_k(x))}_{<\ \varepsilon/3} + \underbrace{d(f_k(x), f_k(z))}_{<\ \varepsilon/3} + \underbrace{d(f_k(z), f(z))}_{<\ \varepsilon/3}$$

$\blacksquare$

So, in particular, if X and Y are both compact metric spaces, then $(C(X,Y), d_{sup})$ is complete.

The Arzela-Ascoli Theorem characterizes the compact subsets of $(\mathcal{C}(X,Y), d_{sup})$

<u>Theorem</u> let X, Y compact.

$\mathcal{F} \subset \mathcal{C}(X,Y)$, d_{sup}

Then $\mathcal{F}$ is compact iff $\mathcal{F}$ is closed and <u>equicontinuous</u>

i.e. $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t.

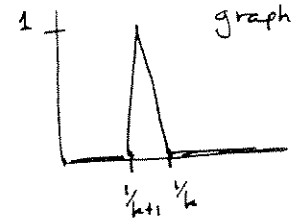
$d(x,z) < \delta \Rightarrow d(f(x), f(z)) < \varepsilon$
$\forall f \in \mathcal{F}$.

[the fcns in $\mathcal{F}$ are uniformly ~~uniformly~~ continuous]

<u>context</u>:

- recall that the closed unit ball in $C([0,1])$ is <u>not</u> compact, e.g. we constructed a sequence $\{f_k\}$, $\|f_k\|_{sup} = 1$, $\|f_k - f_\ell\|_{sup} = 1$ $k \neq \ell$, so that $\{f_k\}$ could have no convergent subsequence.

e.g. graph(f_k).



- however, you proved in a HW (9.2.5 #10) that any compact subset of $C([0,1])$ is equicontinuous, essentially $\Rightarrow$ of Arzela-Ascoli

- Arzela-Ascoli Corollary: let $\{f_k\} \subset \mathcal{C}(X,Y)$ be an equicontinuous sequence. Then $\exists\, \{f_{k_j}\} \to f$ uniformly

  <u>proof</u>: let $\hat{\mathcal{F}} = \overline{\bigcup_k \{f_k\}} = \{f_k\} \cup$ "limit points."

  this family is closed, <u>and</u> equicontinuous, since

  $\forall\, \varepsilon > 0$ $\exists\, \delta > 0$ s.t. $d(x,z) < \delta \Rightarrow d(f_k(x), f_k(z)) < \varepsilon$

  if $\{f_{k_j}\} \to f$ in d_{sup} $\Rightarrow$ deduce $d(f(x), f(z)) \leq \varepsilon$ whenever $d(x,z) < \delta$.

  so, for $\{f_k\} \subset \hat{\mathcal{F}}$

  Arzela Ascoli $\Rightarrow \exists\, \{f_{k_j}\} \to f$ uniformly!

e.g. $\{f_k\} \subset C([0,1])$ equicontinuous <u>and</u> uniformly bounded $(\Rightarrow \{f_k\} \subset C([0,1], [-M,M])$ so A.A. holds)

$\Rightarrow \exists\, \{f_{k_j}\} \to f$

(Can you think of an equicontinuous ~~family~~ seq in $C([0,1])$ without a conv. subseq?)

- Arzela Ascoli is an important tool in DE's, e.g. finding eigenfcns for differential operators proof of Arzela-Ascoli:

text has an interesting proof in §7.6.

we can do a shorter version because we know

$\hat{\mathcal{F}}$ is compact iff $\hat{\mathcal{F}}$ is <u>complete</u> and <u>totally bounded</u>.

$\Updownarrow$

closed, since $\mathscr{C}(X,Y)$ is complete

so all we need to prove is that

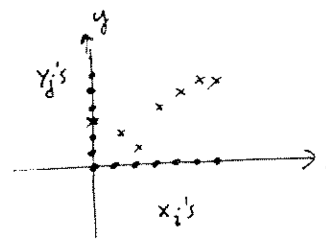
$\hat{\mathcal{F}}$ is equicontinuous iff $\hat{\mathcal{F}}$ is totally bounded

$\Longrightarrow$: let $\varepsilon > 0$

pick δ s.t. $d(x,z) < \delta$

$\Rightarrow d(f(x), f(z)) < \varepsilon/3$

$\forall f \in \hat{\mathcal{F}}$.



let $\{x_i\}_{i=1}^N \subset X$ s.t. $\{y_j\}_{j=1}^M \subset Y$

$X = \bigcup_{i=1}^N B_\delta(x_i)$ $\qquad Y = \bigcup_{j=1}^M B_{\varepsilon/3}(y_j)$

(since X totally bd).

There are exactly M^N distinct mappings from $\{x_i\}_{i=1}^N \to \{y_j\}_{j=1}^M$

if, for one of these mappings χ $\exists f \in \hat{\mathcal{F}}$ with $f(x_i) \in B_{\varepsilon/3}(y_{\chi(i)})$, pick one such f, call it f_χ.

We claim $\hat{\mathcal{F}} \subset \bigcup_\chi B_\varepsilon(f_\chi)$

pf: let $f \in \hat{\mathcal{F}}$. Then $\exists f_\chi$ s.t. $f_\chi(x_i)$ and $f(x_i) \in B_\varepsilon(y_{\chi(i)})$ $\forall i$

let $x \in X$; so $x \in B_\delta(x_i)$, some i $\Rightarrow d(f(x), f_\chi(x))$

$\leq d(f(x), f(x_i)) + d(f(x_i), f_\chi(x_i)) + d(f_\chi(x_i), f_\chi(x))$

$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3$

$\Rightarrow f \in B_\varepsilon(f_\chi)$ ∎

$\Longleftarrow$: let $\varepsilon > 0$

Pick $\{f_k\}_{k=1}^N$ s.t.

$\hat{\mathcal{F}} \subset \bigcup_{k=1}^N B_{\varepsilon/3}(f_k)$.

Each f_k is uniformly cont, so $\exists \delta_k$ s.t. $d(x,z) < \delta_k \Rightarrow d(f_k(x), f_k(z)) < \varepsilon/3$

let $\delta = \min(\delta_1, .. \delta_N)$, and $f \in \hat{\mathcal{F}}$.

Pick f_k s.t. $d(f, f_k) < \varepsilon/3$

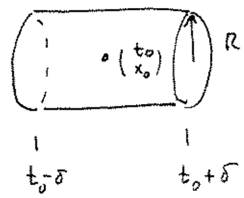
let $d(x,z) < \delta$. Then $d(f(x), f(z)) \leq \underbrace{d(f(x), f_k(x))}_{\vee\ \varepsilon/3} + \underbrace{d(f_k(x), f_k(z))}_{\vee\ \varepsilon/3} + \underbrace{d(f_k(z), f(z))}_{\vee\ \varepsilon/3} < \varepsilon$

Application of Arzela-Ascoli to DE's - section 11.2.

Recall our local $\exists!$ thm using contraction mapping principle:

- Consider

$$\text{IVP} \begin{cases} \frac{d\vec{x}}{dt} = G(t,x) \\ x(t_0) = x_0 \end{cases}$$



with

$G(t,x)$ cont. on Q

$\|G\|_{sup} \le M$ on Q

$\|G(t,x) - G(t,y)\| \le L\|x-y\|$

$\forall\ (t,x),(t,y) \in Q$

$Q = [t_0-\delta, t_0+\delta] \times \overline{B_R(x_0)}$, with $[t_0-\delta, t_0+\delta] \subset \mathbb{R}$ and $\overline{B_R(x_0)} \subset X_{\text{Banach}}$

then we proved

<u>Theorem 1</u> for any $\delta_1 < \min(\delta, \frac{1}{L}, \frac{R}{M})$ $\exists!$ sol'n to IVP on $[t_0-\delta_1, t_0+\delta_1]$.

With $X = \mathbb{R}^k$ (i.e. finite-dim'l), we can use Arzela-Ascoli to prove an existence theorem with weaker conditions on G:

<u>Theorem 2</u>: For $\underline{X = \mathbb{R}^k}$ and $G(t,x)$ cont on Q, let $\|G\|_{sup} = M$ on Q.
Then for $\delta_1 < \min(\delta, \frac{R}{M})$, $\exists$ sol'n $x(t)$ to IVP for $|t-t_0| \le \delta_1$.
(<u>not</u> necessarily unique.)

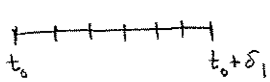
<u>proof</u> Use Euler approximation.

We'll find sol'n on $[t_0, t_0+\delta_1]$.

For $n \in \mathbb{N}$ define the Euler approximation $\vec{x}_n(t)$:

$$\Delta t = \frac{\delta_1}{n}$$

$$t_i = t_0 + i\Delta t \quad , i = 0,1,\ldots n, \quad t_n = t_0 + \delta_1$$



$$\vec{x}_n(t) = x_0 + t\,G(t_0, x_0) \quad 0 \le t \le \Delta t$$

defines $x_n(t_1)$

inductively, if $x_n(t_i)$ is defined,

$$x_n(t) = x_n(t_i) + (t-t_i)G(t_i, x_n(t_i)) \qquad t_i \le t \le t_{i+1}$$

This constructs a piecewise affine, continuous function

$\vec{x}_n(t)$ with $x_n'(t) = G(t_i, x_n(t_i))$, constant for $t_i < t < t_{i+1}$

We have the following estimates:

$|x_n(t) - x_n(t_0)| \le M|t-t_0| < R$ since $|t-t_0| \le \delta_1 < \frac{R}{M}$.

$|x_n(t) - x_n(s)| \le M|t-s|$ (equicontinuity). Also $|x_n(t)| \le |x_0| + R$ (bounded)

In fact, if we write $T_n(s) = t_i$ if $t_i \le s < t_{i+1}$

Then

$$* \qquad x_n(t) = x_0 + \int_{t_0}^{t} \underbrace{x_n'(s)}_{\text{piecewise const}} ds = x_0 + \int_{t_0}^{t} G(T_n(s), x_n(T_n(s)))\, ds$$

Since $\{x_n\}$ is bounded and equicontinuous in $C([t_0, t_0+\delta_1], \mathbb{R}^n)$

$\exists \{x_{n_j}\} \to x(t)$ uniformly in $\|\cdot\|_{sup}$, and $x(t)$ has same equicontinuity est

$T_n(s) \to s$ uniformly as $n \to \infty$ $\left(|T_n(s) - s| \le \frac{\delta_1}{n} \right)$

Deduce $G(T_n(s), x_n(T_n(s))) \to G(s, x(s))$ unif as $n \to \infty$ (Note G is unif cont on Q).

Therefore, let $\{n_j\} \to \infty$ in $*$, deduce

I.E. $$x(t) = x_0 + \int_{t_0}^{t} \underbrace{G(s, x(s))}_{\text{cont!}}\, ds$$

So $$\begin{cases} x'(t) = G(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$ ∎