Math 5210
Friday Jan. 30

HW for Friday 2/6
9.2 8-13, 17, 18
9.3 1-3, 6, 10
Class exercise 1 in today's notes.

- open, closed, interior, closure, bdry, etc
  - Wednesday's notes.

(fix maths at bottom of page 2, should say, if $C_1 \cdots C_k$ are closed so is the union $C_1 \cup C_2 \cup \cdots \cup C_k$.
(arbitrary intersections are closed, but only finite unions)

- In topology any collection of "open" sets $\mathcal{T} = \{\mathcal{O}_\alpha\}$ (subsets of a space X) which satisfies

  (i) $\emptyset, X \in \mathcal{T}$

  (ii) $\{\mathcal{O}_\beta\}_{\beta \in B} \subset \mathcal{T} \Rightarrow \bigcup_{\beta \in B} \mathcal{O}_\beta \in \mathcal{T}$

  (iii) $\{\mathcal{O}_i\}_{i=1..k} \subset \mathcal{T} \Rightarrow \bigcap_{i=1..k} \mathcal{O}_i \in \mathcal{T}$

  is called a topology on X.

  The open sets on a metric space (X,d) are a topology on X.

- If (X,d) is a metric space and $A \subset X$.
  Then (A,d) is also a metric space, and we call it a "subspace".
  not to be confused with a vector subspace!

<u>Theorem</u>: $\mathcal{O} \subset A$ is open in (A,d) ("relatively open", if you're thinking $A \subset X$).
iff $\exists\, \mathcal{U} \subset X$ open s.t. $\mathcal{O} = A \cap \mathcal{U}$

(this theorem is a definition in topology, and then you check that this definition gives a topology on A).

<u>pf</u>: Let $\mathcal{O} \subset (A,d)$ open $\Rightarrow \forall x \in \mathcal{O}\ \exists\ r := r_x$ s.t. $B^A(x, r_x) = \{a \in A \text{ s.t. } d(a,x) < r\} \subset \mathcal{O}$

Let $\mathcal{U} =$

Thus $\mathcal{O} = \bigcup_{x \in \mathcal{O}} B^A_{r_x}(x)$

check: $\mathcal{O} \supset U$

pf: $\Rightarrow$: Let $\mathcal{O} \subset (A,d)$ open

Then $\forall x \in \mathcal{O} \; \exists r_x > 0$ s.t. $B^A_{r_x}(x) = \{a \in A \text{ s.t. } d(a,x) < r_x\} \subset \mathcal{O}$

notice $\quad x \in B^A_{r_x}(x) \subset \mathcal{O}$

so

$$\mathcal{O} = \bigcup_{x \in \mathcal{O}} x \subset \bigcup_{x \in \mathcal{O}} B^A_{r_x}(x) \subset \mathcal{O}$$

i.e. $\mathcal{O} = \bigcup_{x \in \mathcal{O}} B^A_{r_x}(x)$. $\quad$ Let $\mathcal{U} := \bigcup_{x \in \mathcal{O}} \underbrace{B^X_{r_x}(x)}_{\{z \in X \text{ s.t. } d(x,z) < r_x\}}$.

Then $\mathcal{U}$ is open and $\mathcal{O} = \mathcal{U} \cap A$.

$\Leftarrow$ Let $\mathcal{U} \subset X$ open and let $\mathcal{O} = \mathcal{U} \cap A$.

then $x \in \mathcal{O} \Rightarrow r > 0$ s.t. $B^X_r(x) \subset \mathcal{U}_{\text{open}}$

$$\Rightarrow \underbrace{B^X_r(x) \cap A}_{= B^A_r(x).} \subset \mathcal{U} \cap A = \mathcal{O}$$

so $\mathcal{O}$ is open in A!

<u>Example</u>: Let $(X,d) = (\mathbb{R}, |\cdot|)$

Let $A = [0,1]$.

Is $\mathcal{O} = [0, 1/2)$ open in X?

Is $\mathcal{O} = [0, 1/2)$ open in A?

<u>Class exercise 1</u> Consider a metric space (X,d), and subsets.

1a) Show $A \subset B \Rightarrow \mathring{A} \subset \mathring{B}$

$\bar{A} \subset \bar{B}$

$A^c \supset B^c$

1b) Show $(\mathring{A})^c = \overline{(A^c)}$ and $(\bar{A})^c = \mathring{(A^c)}$

1c) Using the operations $\circ$ = interior, $-$ = closure, and $(\;)^c$ = complement, and starting with an arbitrary set A, what is the maximum possible number of distinct sets that can be created by iterating these three operations?

1d) Find a set A (you can do so in $(\mathbb{R}, |\cdot|)$) for which this maximum number is attained!