Metric space completion details (1/28)

<u>Definition</u>: Let (X, d) be a metric space.
Let $(\overline{X}, \overline{d})$ be another metric space such that

(i) $\exists$ an "inclusion" function $i : X \to \overline{X}$ which preserves distances, i.e.

$$\overline{d}(i(x), i(y)) = d(x,y) \quad \forall\, x,y \in X.$$

(Notice this automatically implies $i : X \to \overline{X}$ is 1-1.)

(ii) $i(X)$, i.e. the image of X in $\overline{X}$, is dense in $\overline{X}$

Then $(\overline{X}, \overline{d})$ is called the completion of (X, d)

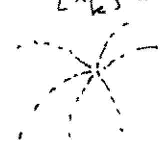
<u>Exercise 1</u> If $(Y, d_1), (Z, d_2)$ are completions of (X, d), show there is a distance preserving bijection $f : (Y, d_1) \to (Z, d_2)$

$$d_2(f(y), f(\tilde{y})) = d_1(y, \tilde{y}) \quad \forall\, y, \tilde{y} \in Y$$

Thus all completions of (X, d) are essentially the same.

<u>Theorem</u> Let (X, d) be a metric space.
The completion $(\overline{X}, \overline{d})$ exists.

<u>proof</u> Define $\overline{X}$ as the collection of equivalence classes of Cauchy sequences $\{x_k\} \subset X$, where $\{x_k\} \equiv \{y_k\}$ iff $\lim_{k\to\infty} d(x_k, y_k) = 0$



(0) $\equiv$ defines an equivalence relation:

(i) $\{x_k\} \equiv \{x_k\}$ $\qquad d(x_k, x_k) = 0 \to 0$ as $k \to \infty$

(ii) $\{x_k\} \equiv \{y_k\}$ iff $\{y_k\} \equiv \{x_k\}$ $\qquad d(x_k, y_k) \to 0 \Rightarrow d(y_k, x_k) \to 0$ since d is symmetric

(iii) $\{x_k\} \equiv \{y_k\}$, $\{y_k\} \equiv \{z_k\} \Rightarrow \{x_k\} \equiv \{z_k\}$: $\qquad d(x_k, z_k) \le d(x_k, y_k) + d(y_k, z_k)$

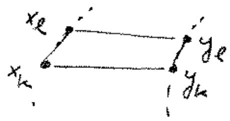
Thus the equivalence classes partition the collection of all Cauchy sequences, and $\overline{X}$ exists as a set.

(1) Let $\overline{x} = [\{x_k\}]$, $\overline{y} = [\{y_k\}] \in \overline{X}$.

Define $\overline{d}(\overline{x}, \overline{y}) = \lim_{k\to\infty} d(x_k, y_k)$

$\overline{d}$ is well-defined, and defines a distance on $\overline{X}$

1 a) $\lim_{k\to\infty} d(x_k, y_k)$ exists, because the sequence of real numbers $\{d(x_k,y_k)\}$ is Cauchy.



In fact, the triangle inequality implies

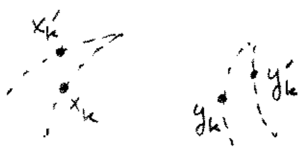
$$d(x_k,y_k) \le d(x_k,x_\ell) + d(x_\ell,y_\ell) + d(y_\ell,y_k)$$

i.e. $\quad d(x_k,y_k) - d(x_\ell,y_\ell) \le d(x_k,x_\ell) + d(y_k,y_\ell)$

also $\quad d(x_\ell,y_\ell) - d(x_k,y_k) \le \quad$ " $\quad$ "

so $$|d(x_k,y_k) - d(x_\ell,y_\ell)| \le \underset{\substack{\downarrow k,\ell\to\infty \\ 0}}{d(x_k,x_\ell)} + \underset{\substack{\downarrow k,\ell\to\infty \\ 0}}{d(y_k,y_\ell)}$$ ∎

1b) Well-defined: If $\{x_k'\} \equiv \{x_k\}$
& $\{y_k'\} \equiv \{y_k\}$
then (estimate as above, using the "quadrulateral inequality")



$$|d(x_k,y_k) - d(x_k',y_k')| \le \underset{\substack{\downarrow \\ 0}}{d(x_k,x_k')} + \underset{\substack{\downarrow \\ 0}}{d(y_k,y_k')}$$

so $\{d(x_k,y_k)\}$ and $\{d(x_k',y_k')\}$ have the same limit.

1c) $\bar{d}$ is a metric: Let $\bar{x} = [\{x_k\}]$, $\bar{y} = [\{y_k\}]$, $\bar{z} = [\{z_k\}]$

(1) $\bar{d}(\bar{x},\bar{y}) = \lim_{k\to\infty} d(x_k,y_k) \ge 0$ since each $d(x_k,y_k) \ge 0$.

$d(\bar{x},\bar{y}) = 0$ iff $\lim_{k\to\infty} d(x_k,y_k) = 0$ iff $\{x_k\} \equiv \{y_k\}$ iff $\bar{x} = \bar{y}$.

(2) $\bar{d}(\bar{x},\bar{y}) = \bar{d}(\bar{y},\bar{x})$ clear

(3) $$d(x_k,z_k) \le d(x_k,y_k) + d(y_k,z_k)$$
$$\downarrow k\to\infty$$
$$\bar{d}(\bar{x},\bar{z}) \le \bar{d}(\bar{x},\bar{y}) + \bar{d}(\bar{y},\bar{z})$$

since non-strict inequalities are preserved under limits.

(2) Define the inclusion map $i : X \to \bar{X}$

$$i(x) := [\{x,x,x,\dots\}]$$

then $\bar{d}(i(x), i(y)) = \lim_{k\to\infty} d(x,y) = d(x,y)$

(3) If $\bar{x} = [\{x_k\}] \in \bar{X}$, then $i(x_k) \to \bar{x}$ in $(\bar{X}, \bar{d})$
so $i(X)$ is dense in $\bar{X}$.

pf: $i(x_k) = [\{x_k, x_k, x_k, \cdots\}]$

$$\bar{d}(i(x_k), \bar{x}) = \lim_{j\to\infty} d(x_k, x_j)$$

Let $n \in \mathbb{N}$. $\exists m$ s.t. $j, k \geq m \Rightarrow d(x_k, x_j) < \frac{1}{n}$

$$\Rightarrow \lim_{j\to\infty} d(x_k, x_j) \leq \frac{1}{n}$$

$$\Rightarrow \bar{d}(i(x_k), \bar{x}) \leq \frac{1}{n}$$

i.e. $k \geq m \Rightarrow \bar{d}(i(x_k), \bar{x}) \leq \frac{1}{n}$.

Thus $\lim_{k\to\infty} \bar{d}(i(x_k), \bar{x}) = 0$. ■

(4) $(\bar{X}, \bar{d})$ is complete.

pf: Let $\{\bar{x}_k\} \subset \bar{X}$ Cauchy.

Pick $x_k \in X$ s.t. $\bar{d}(i(x_k), \bar{x}_k) < \frac{1}{k}$ $\quad k = 1, 2, \ldots$
using that $i(X)$ is dense in $\bar{X}$, which follows from (3).

then
$$\underbrace{\bar{d}(i(x_k), i(x_\ell))}_{= d(x_k, x_\ell)} \leq \underbrace{\bar{d}(i(x_k), \bar{x}_k)}_{< \frac{1}{k}} + \underbrace{\bar{d}(\bar{x}_k, \bar{x}_\ell)}_{\xrightarrow[\text{Cauchy}]{k,\ell\to\infty} 0} + \underbrace{\bar{d}(\bar{x}_\ell, i(x_\ell))}_{< \frac{1}{\ell}}$$

$\to 0$ as $k, \ell \to \infty$.

Thus $\{x_k\} \subset X$ is Cauchy, and $\bar{z} := [\{x_k\}] \in \bar{X}$.

We claim $\{\bar{x}_k\} \to \bar{z}$:

$$\bar{d}(\bar{x}_k, \bar{z}) \leq \underbrace{\bar{d}(\bar{x}_k, i(x_k))}_{< \frac{1}{k} \text{ by construction}} + \underbrace{\bar{d}(i(x_k), \bar{z})}_{\to 0 \text{ by (3)}} \quad \xrightarrow{\text{as } k\to\infty} 0$$

■