Math 5210
Wednesday Jan 28

- On HW for Friday, postpone 9.2 #7, 9, but do all other problems.
- As mentioned yesterday, the inequality you're chasing in class exercise 1a is

$$t^\lambda \le (1-\lambda) + \lambda t \qquad 0<\lambda<1,\ t>0$$

which follows from

$$\varphi(t) = (1-\lambda) + \lambda t - t^\lambda \ge 0$$

so it suffices to show φ has a minimum value of 0 at $t=1$

- Fill in the proof that every (X,d) is dense in a (unique) natural completion, the complete metric space $(\bar{X},\bar{d})$ ~ Tuesday notes.

- §9.2 Open and closed sets in metric spaces.

Let (X,d) a metric space.

$B_r(x) := \{y \in X \text{ s.t. } d(x,y) < r\}$

(so-called open ball of radius r centered at x).

Def $\mathcal{O} \subset X$ is open iff $\forall x \in \mathcal{O}\ \exists\ r>0$ s.t. $B_r(x) \subset \mathcal{O}$

(or $\forall x \in \mathcal{O}\ \exists\ m \in \mathbb{N}$ s.t. $B_{1/m}(x) \subset \mathcal{O}$).

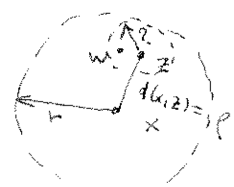
Lemma Let $r>0$, $x \in X$. Then $B_r(x)$ is open!

pf Let $z \in B_r(x)$

$\Rightarrow d(x,z) := \rho < r$

Let $\rho_z := r - \rho > 0$

Claim $B_{\rho_z}(z, \rho_z) \subset B_r(x)$



pf $d(w,z) < \rho_z = r - \rho$

$\Rightarrow d(x,w) \le \underbrace{d(x,z)}_{=\rho} + \underbrace{d(z,w)}_{< r-\rho} < r \quad \Rightarrow w \in B_r(x)$ ∎

Theorem (1) $\bigcup_{\alpha \in A} \mathcal{O}_\alpha$ is open if each $\mathcal{O}_\alpha$ open

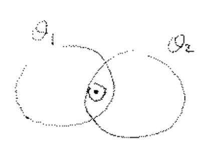
(2) $\mathcal{O}_1 \cap \mathcal{O}_2 \cap \dots \cap \mathcal{O}_k$ is open if each $\mathcal{O}_j$ is open

Arbitrary unions of open sets are open
finite intersections of open sets are open

pf (1) Let $x \in \mathcal{O}_{\alpha_0}$ some α_0

$\Rightarrow \exists\ r>0$ s.t. $B_r(x) \subset \mathcal{O}_{\alpha_0} \subset \bigcup_{\alpha \in A} \mathcal{O}_\alpha$ ∎

(2) Let $x \in \mathcal{O}_1 \cap \dots \cap \mathcal{O}_k$.



$\Rightarrow \exists\ r_j > 0$ s.t. $B(x, r_j) \subset \mathcal{O}_j$

Let $r = \min(r_1, \dots, r_k) > 0$

$\Rightarrow B(x,r) \subset B(x,r_j) \subset \mathcal{O}_j \quad \forall j$

$\Rightarrow B(x,r) \subset \mathcal{O}_1 \cap \dots \cap \mathcal{O}_k$ ∎

Def x is a limit point of $A \subset X$

iff $\forall r>0 \; \exists \, a \in A, \; a \neq x$ s.t. $d(a,x) < r$

$\downarrow$ $n \in \mathbb{N}$ $\qquad$ $\downarrow$ $1/n$

iff $\exists \, \{a_k\} \subset A, \; a_k \neq x$, $d(x, a_k) < 1/k$

iff $\exists \, \{b_k\} \subset A, \; b_k \neq x$, $\{b_k\} \to x$.

Examples

What are the limit points of $\{q \in \mathbb{Q} \text{ s.t. } 0<q<1\} \subset \mathbb{R}$?

What are the limit points of $\mathbb{Z} \subset \mathbb{R}$?

Definition C is closed iff it contains all its limit points (i.e. you can't leave C by taking the limit of a Cauchy sequence)

Theorem C is closed iff its complement C^c is open

proof: $\Rightarrow$ Let C be closed, let $x \in C^c$.

If $\nexists \, r$ s.t. $B_r(x) \subset C^c$

then $\forall r>0 \; \exists \, c \in C, \; c \neq x$ s.t. $d(x,c) < r$

$\Rightarrow$ x is a limit point of C

$\Rightarrow x \in C$ since C is closed. $\times$

Thus $\exists \, r>0$ s.t. $B_r(x) \subset C^c$ $\blacksquare$

$\Leftarrow$ Let C^c be open. Let x be a limit point of C.

If $x \notin C$ then $x \in C^c$ so $\exists \, r>0$ s.t. $B_r(x) \subset C^c$

$\Rightarrow$ x is not a limit pt.

Thus $x \in C$. $\blacksquare$

Cor. If each C_α is $\{C_\alpha\}$ is closed then so is $\bigcap_\alpha C_\alpha$.

If $C_1, \dots C_k$ are closed, then so is $C_1 \cap C_2 \cap \dots \cap C_k$

$\leftarrow$ proof is just de Morgan's laws. (Which you should check!)

$$\left[\bigcap_\alpha C_\alpha\right]^c = \bigcup_\alpha C_\alpha^c \text{ is open}$$

$$\left(\bigcup_{i=1}^k C_i\right)^c = \bigcap_{i=1}^k C_i^c \text{ is open.}$$

$\boxed{\text{Closed sets are closed w.r.t. arbitrary intersections and finite unions}}$

Def $\mathring{A}$, the interior of A, is $\{x \in A \text{ s.t. } \exists r > 0 \text{ s.t. } B_r(x) \subset A\}$

$\bar{A}$, the closure of A, is $A \cup \{\text{limit points of } A\}$

∂A, the boundary of A, is $\left\{ x \in X \text{ s.t. } \forall r > 0,\ B_r(x) \cap A \neq \emptyset \text{ and } B_r(x) \cap A^c \neq \emptyset \right\}$.

Theorem $\mathring{A}$ is open. If $\Theta \subset A$ is open, then $\Theta \subset \mathring{A}$. ($\mathring{A}$ is the largest open subset of A)
A is open iff $A = \mathring{A}$

$\bar{A}$ is closed. If $A \subset C$ and C is closed, then $\bar{A} \subset C$ ($\bar{A}$ is the smallest closed set containing A)
A is closed iff $A = \bar{A}$

Let $A \subset X$.
Every $x \in X$ is in exactly one of $\mathring{A}$, $(\mathring{A^c})$, or ∂A, i.e. these 3 subsets partition X.

Examples Let $A = B_r(x) \subset \mathbb{R}^2$ $(= (\mathbb{R}^2, \|\ \|_2))$
What are $\mathring{A}, \bar{A}, \partial A$?

Let $A = \{q \in \mathbb{Q},\ 0 < q < 1\} \subset \mathbb{R}$
What are $\mathring{A}, \bar{A}, \partial A$?