Math 5210
Tuesday Jan 27
§9.1 : Complete metric spaces

<u>Def</u> (X,d) is complete iff $\{x_k\} \subset X$ Cauchy $\Rightarrow \exists x \in X$ s.t. $\{x_k\} \to x$

A complete normed vector space is called a <u>Banach Space</u>

A complete inner product space is called a <u>Hilbert Space</u>

We already know $(\mathbb{R}, |\cdot|)$ is complete

<u>Theorem 1</u> $\mathbb{R}^n$ is complete with respect to any of the $\|\cdot\|_p$, $1 \le p \le \infty$.

<u>lemma</u>: Let $1 \le p, q \le \infty$. Then $\|x\|_p \le n \|x\|_q$.

Thus $\mathbb{R}^n$-distances measured with p-norms are uniformly comparable, and $\left\{\begin{matrix}\text{convergence} \\ \text{or} \\ \text{Cauchy}\end{matrix}\right\}$ in one p-norm implies the same, in every q-norm.

<u>proof</u> $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \le \left(\sum_{i=1}^n \|x\|_\infty^p\right)^{1/p} = n^{1/p}\|x\|_\infty \le n\|x\|_\infty$

$1 \le p < \infty$.

Thus, for $1 \le p \le \infty$, $\|x\|_p \le n\|x\|_\infty$.

but $\|x\|_\infty = \max\{|x_i|, i = 1 \cdots n\}$

$\le \left(\sum |x_i|^q\right)^{1/q}$

$= \|x\|_q$ $\quad 1 \le q < \infty$. also $q = \infty$.

so $\boxed{\|x\|_p \le n\|x\|_q}$ ∎

<u>proof of Thm 1</u>

Let $\{\vec{x}_k\}$ Cauchy in $\|\cdot\|_p$. Write $\vec{x}_k = \begin{bmatrix} x_{k,1} \\ x_{k,2} \\ \vdots \\ x_{k,n}\end{bmatrix}$.

$\Rightarrow \{x_k\}$ Cauchy in $\|\cdot\|_\infty$. (by lemma)

i.e. $\forall \varepsilon\ \exists\ m$ s.t. $k, l \ge m \Rightarrow \|x_k - x_l\|_\infty < \varepsilon$

$\Rightarrow |x_{k,i} - x_{l,i}| < \varepsilon$ each $i = 1, \cdots n$

$\Rightarrow$ each coord $\{x_{k,i}\}$ is Cauchy in $\mathbb{R}$

$\Rightarrow \exists\ z_i \in \mathbb{R}$ s.t. $\{x_{k,i}\} \to z_i$ as $k \to \infty$.

Let $z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$

Then $\|x_k - z\|_1 = \sum_{i=1}^n |x_{k,i} - z_i| \longrightarrow 0$ as $k \to \infty$

$\Rightarrow \|x_k - z\|_p \to 0$, i.e. $\{x_k\} \to z$ in p-norm distance.
(lemma) ∎

<u>Theorem 2</u> $\quad (C([a,b]), \|\cdot\|_\infty)$ <u>is</u> complete.

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)| = \max_{x\in[a,b]} |f(x)| < \infty \quad \text{for continuous functions } f: [a,b] \to \mathbb{R}.$$

Recall, f cont at x_0 iff $\forall n \in \mathbb{N}\ \exists\ m$ s.t. $|x-x_0| < \frac{1}{m} \Rightarrow |f(x)-f(x_0)| < \frac{1}{n}$

(iff $\forall \varepsilon > 0\ \exists\ \delta > 0$ s.t. $|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon$).

f cont on $[a,b]$ iff f cont at x_0, $\forall x_0 \in [a,b]$.

<u>Step 1</u> find what you hope the limit is – just like page 1:

Let $\{f_k\}$ Cauchy, let $x_0 \in [a,b]$. $\qquad |f_k(x_0) - f_\ell(x_0)| \leq \|f_k - f_\ell\|_\infty$

So $\{f_k\}$ Cauchy in $C([a,b]) \Rightarrow \{f_k(x_0)\}$ Cauchy in $\mathbb{R}$.

Define $f(x_0) := \lim_{k\to\infty} f_k(x_0)$.

<u>Step 2</u> Show $\{f_k\} \to f$ in $\|\cdot\|_\infty$.

Let $n \in \mathbb{N}$. $\exists\ m$ s.t. $k, \ell \geq m \Rightarrow \|f_k - f_\ell\|_\infty < \frac{1}{n}$

$$\Rightarrow |f_k(x) - f_\ell(x)| < \tfrac{1}{n} \quad \forall x \in [a,b].$$

$$\Rightarrow$$

$$f_k(x) - \tfrac{1}{n} < f_\ell(x) < f_k(x) + \tfrac{1}{n}$$

$$\lim_{\ell\to\infty}: \qquad f_k(x) - \tfrac{1}{2} \leq f(x) \leq f_k(x) + \tfrac{1}{n}$$

$$\text{i.e.} \quad |f_k(x) - f(x)| \leq \tfrac{1}{n} \qquad \forall x \in [a,b].$$

$$k \geq m \Longrightarrow \|f_k - f\|_\infty \leq \tfrac{1}{n} \qquad \blacksquare$$

<u>Step 3</u> $f \in C([a,b])$

i.e. uniform limits of continuous functions are continuous

Let $x_0 \in [a,b]$, $n \in \mathbb{N}$.

Pick k s.t. $\|f - f_k\| < \frac{1}{n}$

$f_k \in C([a,b])$ so pick $\delta > 0$ s.t. $|x-x_0| < \delta \Rightarrow |f_k(x) - f_k(x_0)| < \frac{1}{n}$

Then $|x - x_0| < \delta$

$$\Rightarrow |f(x) - f(x_0)| = |f(x) - f_k(x) + f_k(x) - f_k(x_0) + f_k(x_0) - f(x_0)|$$

$$\leq |f(x) - f_k(x)| + |f_k(x) - f_k(x_0)| + |f_k(x_0) - f(x_0)|$$

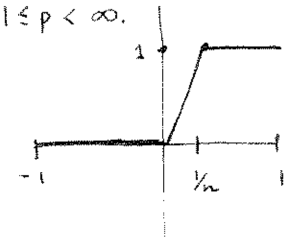
$$< \tfrac{1}{n} + \tfrac{1}{n} + \tfrac{1}{n} = 3/n$$

$\blacksquare$

<u>Theorem 3</u> $(C([a,b]), \|\cdot\|_p)$ is <u>not</u> complete for $1 \le p < \infty$.

<u>proof</u> (See HW problem #3 in 9.2)

e.g. $\{f_n\} \subset C([-1,1])$ with graphs



is Cauchy in $\|\cdot\|_p$

for $1 \le p < \infty$!

Showing this gets you started on HW problem!

by the way, the space of piecewise cont fns isn't complete either!

On the other hand,

<u>Theorem 4</u> Let (X, d) be a metric space. We can "complete" X to its completion $(\bar{X}, \bar{d})$ as follows:

Let $\{x_k\}$, $\{y_k\}$ Cauchy sequences in X.

$\{x_k\} \equiv \{y_k\}$ iff $\{d(x_k, y_k)\} \to 0$.

This is an equivalence relation.

Write $[\{x_k\}]$ for the equivalence class of Cauchy sequences equivalent to $\{x_k\}$.

Let $\bar{X}$ = the collection of equivalence classes of Cauchy sequences in X.

Defined $\bar{d}(x,y) = \lim_{k \to \infty} d(x_k, y_k)$ where $\{x_k\} \in [x]$, $\{y_k\} \in [y]$.

- $\bar{d}$ is well-defined (and finite!)

- $\bar{d}$ defines a distance on $\bar{X}$

- the map $i : X \to \bar{X}$, $i(x) = [\{x\}]$ is a distance preserving injection

- If $x = [\{x_k\}]$ $(\{x_k\} \subset X)$
  then $\{x_k\} \to x$ in $(\bar{X}, \bar{d})$
  (So X is dense in $\bar{X}$)

- $(\bar{X}, \bar{d})$ is complete!

Thus, for $1 \le p < \infty$, $C([a,b], \|\cdot\|_p)$ has an abstract completion.
Is there a way to concretely understand the elements (equivalence classes)
of this completion? ~ Yes, see measure theory and the Lebesgue integral!