Math 5210
Friday Jan 23

- Start and finish discussion of limit points, pages 2-4 Wednesday notes
- Extended reals remark below
- Begin Chapter 9: Metric spaces

Homework for Friday 1/30

3.1.3 # 1, 2, 4, 5, 7, 9, 11
9.1.4 # 4, 7, 10, 14
9.2.5 # 2, 3, 7, 9, 14, 15

do this for $d_p(f,g) = \|f\|_p$, all $1 \le p < \infty$.

Also class exercise 1abc.
(see today's notes)

Extended reals: $\mathbb{R} \cup \{+\infty, -\infty\} = \overline{\mathbb{R}}$

$x \in \mathbb{R} \Rightarrow -\infty < x < \infty$.

$\{x_k\} \to +\infty$ iff

$\forall n \in \mathbb{N}\ \exists m$ s.t. $k \geq m \Rightarrow x_k > n$

(similar for $x \to -\infty$).

<u>Theorem</u>: Let $\{x_k\} \nearrow$. Then either

(a) $\{x_k\}$ is bounded and $\{x_k\} \to \sup\{x_k\}$.

or (b) $\{x_k\} \to +\infty$.

<u>pf</u> If $\{x_k\}$ is unbounded, then $\nexists\, n \in \mathbb{N}$ s.t. $x_k \le n\ \forall k$

so $\forall n \in \mathbb{N}\ \exists\, m$ s.t. $x_m > n$

so $k \geq m \Rightarrow x_k > n$

$\Rightarrow \{x_k\} \to +\infty$

(similar thm for $\{x_k\} \searrow$.)

- In the extended reals every set has upper and lower bounds, unique l.u.b's and g.l.b's.
- Every monotone sequence has a limit : That's the Theorem above!

- And the sequence theorem from Wed notes still holds, for arbitrary sequences:

Let $\{x_k\} \subset \overline{\mathbb{R}}$. Then

a) $\underbrace{\liminf_{k\to\infty} \{x_k\}}_{=x}$, $\underbrace{\limsup_{k\to\infty} \{x_k\}}_{=X}$ are limit points.

b) if z is any limit point, then $x \le z \le X$

c) $\lim_{k\to\infty} x_k$ exists (in $\overline{\mathbb{R}}$) iff $x = X$.

Chapter 9...

(X, d) is a <u>metric space</u> if X is a set
and $d: X \times X \to \mathbb{R}$ is a non-negative fcn satisfying $\forall x, y, z$

1) $d(x,y) \geq 0$, equality iff $x = y$
2) $d(x,y) = d(y,x)$ symmetry
3) $d(x,z) \leq d(x,y) + d(y,z)$ triangle inequality.

- $(\mathbb{R}, d(x,y) = |x-y|)$ is a metric space
- $(\mathbb{R}^n, d(x,y) = |x-y| = \|x-y\|_2 = \left[\sum_i (x_i - y_i)^2\right]^{1/2})$ is a metric space.

Recall, V is a <u>vector space</u> if the set has an addition operation $+$, and (usually real) scalar multiplication, so that

- V closed wrt $+$ and scalar mult
- $v + w = w + v$
  $(v+w) + z = v + (w+z)$
- $t(v+w) = tv + tw$
- $1 \cdot v = v$, $0 + v = v$

$(V, \|\cdot\|)$ is a <u>normed vector space</u> and $\|\cdot\|$ is a norm iff $\forall$ v, w, z in vector space V, $t \in \mathbb{R}$

1) $\|v\| \geq 0$, equality iff $v = 0$ (positivity)
2) $\|tv\| = |t| \, \|v\|$ (homogeneity)
3) $\|v+w\| \leq \|v\| + \|w\|$ Δ ineq.

- Every normed vector space has a metric space structure, if we define $d(x,y) = \|x-y\|$
  Check!

<u>Examples</u>

$(\mathbb{R}^n, \|\cdot\|_p)$

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

$1 \leq p \leq \infty$.
for $p = \infty$, $\|x\|_\infty := \max_i |x_i|$

$(C[a,b], \|\cdot\|_p)$

$\{f : [a,b] \to \mathbb{R} \text{ s.t. } f \text{ is continuous}\}$

$$\|f\|_p = \left(\int_a^b |f(t)|^p \, dt\right)^{1/p}$$

$1 \leq p \leq \infty$.
for $p = \infty$, $\|f\|_\infty := \max_{t \in [a,b]} |f(t)|$

We can check $p = 1, \infty$ easily:

$\boxed{\text{actually, for } p \neq 1, 2, \infty \text{ the } \Delta \text{ inequality is quite subtle, see Exercise 1}}$

An even more specialized structure is an inner product space:

$(V, \langle \cdot , \cdot \rangle)$ is a real inner product space iff V is a vector space with an inner product satisfying $\forall\, v, w, z \in V$, $s, t \in \mathbb{R}$

1) $\langle v, v\rangle \geq 0$, equality iff $v = 0$
2) $\langle v, w\rangle = \langle w, v\rangle$ symmetry
3) bilinearity: $\langle sv + tz, w\rangle = s\langle v, w\rangle + t\langle z, w\rangle$

<u>Thm</u> If $\langle \cdot , \cdot \rangle$ is a real inner product, then $\|v\| := \langle v, v\rangle^{1/2}$ defines a norm.

<u>lemma</u> $|\langle v, w\rangle| \leq \|v\|\|w\|$ (Cauchy-Schwarz)

<u>pf</u>: $\langle v + tw, v + tw\rangle \geq 0 \quad \forall t$ (= 0 iff v, w dependent) for some t

$$p(t) = t^2\langle w, w\rangle + 2t\langle v, w\rangle + \langle v, v\rangle \geq 0 \quad \forall t$$

so $p(t)$ has 0 or 1 real root $\Rightarrow$ discrim < 0 or $= 0$

$$\Rightarrow 4\left(\langle v, w\rangle^2 - \langle v, v\rangle\langle w, w\rangle\right) < 0 \quad (\text{or} = 0)$$

$$\Rightarrow \langle v, w\rangle^2 \leq \|v\|^2\|w\|^2 \quad (\text{or} <). \quad \blacksquare$$

pf of Thm:

$$\|v + w\|^2 = \langle v + w, v + w\rangle$$
$$= \|v\|^2 + 2\langle v, w\rangle + \|w\|^2$$
$$\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 \quad \text{(C.S. lemma)}$$
$$= (\|v\| + \|w\|)^2 \quad \blacksquare$$

(and = iff v, w are positive scalar mults of eachother.)

<u>examples</u>

$\mathbb{R}^n$, dotprod. $\bullet$

also $\mathbb{R}^n, \langle\ ,\ \rangle$ where $\langle x, y\rangle := x \cdot Ay$ where A is symmetric, (so $x \cdot Ay = Ax \cdot y$) and positive definite ($x^TAx > 0$ $\forall x \neq 0$)

$= x^TAy = y^TAx$

$(C[a,b], \langle\ ,\ \rangle)$

$$\langle f, g\rangle = \int_a^b f(t)g(t)\,dt$$

orthogonality is a very useful tool in inner product spaces — think Gram-Schmidt, Fourier series etc... ortho-normal bases,

$v \perp w$ iff $\langle v, w\rangle = 0$

<u>Definitions</u> Let (X, d) be a metric space. Fill in these definitions

$\{x_k\}$ <u>Cauchy</u> iff

(i) $\frac{1}{n}$ version

$\Updownarrow$

(ii) ε-version

$\{x_k\} \to x$ iff

Make life easier version(s):

Prove all convergent sequences are Cauchy:

Prove limits are unique: i.e. $\{x_k\} \to x$, $\{x_k\} \to y \Rightarrow x = y$.

<u>Definition</u>: (X, d) is <u>complete</u> iff $\{x_k\} \subset X$ Cauchy $\Rightarrow \exists\, x \in X$ s.t. $\{x_k\} \to x$

A complete normed vector space is a <u>Banach Space</u>

A complete inner product space is called a <u>Hilbert Space</u>

(We will be especially interested in ∞-dim'l Banach & Hilbert spaces)

On Monday we will prove

<u>Theorem</u> $\mathbb{R}^n$ is complete with respect to any of the $\|\cdot\|_p$, $1 \le p \le \infty$. In fact, for any $1 \le p, q \le \infty$

$$\|x\|_p \le n \|x\|_q$$

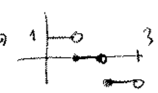
so $\mathbb{R}^n$-distances measured with p-norms are uniformly comparable, and convergence in one p-norm implies convergence in all the others.

Thus (using $p = \infty$), $\{x_k\} \subset \mathbb{R}^n$ is Cauchy or convergent iff each component sequence $\{x_k^i\}$ is $(i = 1, 2, \ldots n)$, and this is why $\mathbb{R}^n$ is complete.

Exercise 1 Prove that the p-norm $\|\cdot\|_p$ satisfies the triangle inequality, $1<p<\infty$.

note: Use function notation, since this includes the case of $\mathbb{R}^n$ if you identify $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ with the integrable bounded fcn on $[0,n]$

$$f_{\vec{x}}(t) = \begin{cases} x_1 & 0 \le t < 1 \\ x_2 & 1 \le t < 2 \\ \vdots & \\ x_n & n-1 \le t < n \end{cases}$$

e.g. $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rightsquigarrow$ 

then
$$\|x\|_p = \|f_{\vec{x}}\|_p .$$

(1a) Prove the generalized geometric-arithmetic mean inequality:

$$\boxed{\begin{matrix}\alpha,\beta \ge 0 \\ 0<\lambda<1\end{matrix} \Rightarrow \alpha^{\lambda}\beta^{1-\lambda} \le \lambda\alpha + (1-\lambda)\beta \qquad (\text{equality iff } \alpha=\beta)}$$

hint: Consider, for $t>0$, $\varphi(t) = (1-\lambda) + \lambda t - t^{\lambda}$.
Use Calculus ($\varphi'(t)$) to show that for $t>0$ φ has a maximum value of 0, at $t=1$.
Then, for $\beta \neq 0$, manipulate $\varphi(\frac{\alpha}{\beta}) \le 0$ to deduce result
check $\beta=0$ separately.

(1b) Prove the generalized Cauchy-Schwarz inequality (called ~~Min~~ Hölder ineq.)

$$\boxed{\int_a^b |f(t)g(t)|\,dt \le \|f\|_p \|g\|_q} \qquad \text{where } \frac{1}{p}+\frac{1}{q}=1 \quad (\text{i.e. } q=\tfrac{p}{p-1})$$

hint: First deduce Hölder in case $\|f\|_p = \|g\|_q = 1$ by deriving
$$|f(t)|\,|g(t)| \le \frac{1}{p}|f(t)|^p + \frac{1}{q}|g(t)|^q$$
from (1a), and integrating.
Derive the general case by considering $\frac{f}{\|f\|_p}$, $\frac{g}{\|g\|_q}$ and applying what you just derived.

(1c) Finally, prove $\|f+g\|_p \le \|f\|_p + \|g\|_p$ by starting with

$$\|f+g\|_p^p = \int_a^b |f(t)+g(t)|^p\,dt = \int_a^b |f(t)+g(t)|\,|f(t)+g(t)|^{p-1}\,dt$$
$$\le \int_a^b |f(t)|\,|f(t)+g(t)|^{p-1} + |g(t)|\,|f(t)+g(t)|^{p-1}\,dt$$

and Höldering these last two inequalities in the only way that would make sense!