Addendum to Wed 1/21 notes

- The proof of Cauchyness for $\{x_k\}, \{y_k\}$ can be shortened from the geometric series argument, in the $\exists$ l.u.b. theorem, pages 1,2.
(Thanks to Ryan T. for pointing this out.)

In the subdivision argument we have

$$\{x_k\} \nearrow$$
$$\{y_k\} \searrow$$
$$y_k = x_k + \frac{L}{2^{k-1}}$$

for $l \geq k$, $x_k \leq x_l < y_k$

$$\Rightarrow |x_l - x_k| = x_l - x_k < y_k - x_k = \frac{L}{2^{k-1}}$$

Thus as $k, l \to \infty$, $|x_l - x_k| \to 0$, i.e. $\{x_k\}$ Cauchy.

(precise: Let $n>0$. pick m s.t. $\frac{L}{2^{m-1}} < \frac{1}{n}$
then $k, l \geq m \Rightarrow |x_l - x_k| < \frac{1}{n}$.)

Similarly $\{y_k\}$ Cauchy.

- Using the fact that sets which are bounded above have l.u.b.'s lets you shorten proofs like $\sqrt{2}$ exists:

Let $E = \{x \geq 0 \text{ s.t. } x^2 \leq 2\}$. E is bounded above, e.g. by $y=2$, since $x \geq 2 \Rightarrow x^2 \geq 4 > 2$

Let $y = \text{l.u.b.}(E)$.

We claim $y^2 = 2$:

if $y^2 < 2$ then $(y+\frac{1}{n})^2 = y^2 + \frac{2y}{n} + \frac{1}{n^2} < 2$ for n large
so $x = y + \frac{1}{n} \in E$, so y is not an u.b.

if $y^2 > 2$ then $(y - \frac{1}{n})^2 > 2$ for n large
so $y - \frac{1}{n}$ is an u.b., so y is not the l.u.b.

thus $y^2 = 2$ ∎