Math 5210
Wednesday Jan 20 §3.1

HW for Friday is
2.2 & 2.3.
Postpone 3.1

- Finish pages 2,3 Tuesday.

Page 3 technique gets used a lot! (See below!)

let $E \subset \mathbb{R}$, $E \neq \emptyset$

<u>Definition</u> $y \in \mathbb{R}$ is an <u>upper bound for E</u> iff $x \leq y \ \forall x \in E$
" " <u>lower</u> " " $y \leq x \ \forall x \in E$

<u>Definition</u> E is <u>bounded above</u> iff $\exists$ an upper bound y for E (iff $\exists N$ s.t. $x \leq N \ \forall x \in E$)
" " <u>below</u> " " <u>lower</u> " " (- - - - - $x \geq -N$ $\cdots$)

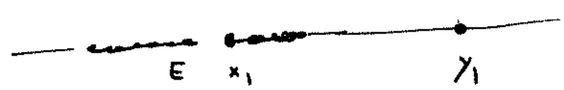
<u>Def</u> E is <u>bounded</u> iff $\exists$ an upper bd & a lower bd for E (iff $\exists N$ s.t. $|x| \leq N \ \forall x \in E$)

<u>Theorem</u>: If E is bounded above $\exists!\ y \in \mathbb{R}$ which is an upperbound for E and such that every other upperbound $y' \geq y$.

This y is called the <u>least upper bound</u> (lub) or <u>supremum</u> (sup) of E.
(Similarly, if E is bounded below it has a unique greatest lower bound (glb) or infimum (inf))

<u>proof</u>: This mimics to $\sqrt{2}$ proof!

Pick $x_1 < y_1$ s.t. x_1 <u>not</u> u.b., y_1 is u.b.



if $\frac{x_1+y_1}{2}$ is an upper bound, $y_2 = \frac{x_1+y_1}{2}$, $x_2 = x_1$

if not, $x_2 = \frac{x_1+y_1}{2}$, $y_2 = y_1$

Inductively, given $x_k < y_k$ x_k not u.b., y_k is u.b.

if $\frac{x_k+y_k}{2}$ is u.b. $x_{k+1} := x_k$, $y_{k+1} := \frac{x_k+y_k}{2}$

else $x_{k+1} = \frac{x_k+y_k}{2}$, $y_{k+1} = y_k$

<u>Inductive properties</u>

x_k not u.b., y_k u.b.

$x_k < y_k$; $y_k = x_k + \frac{L}{2^{k-1}}$ for $L = y_1 - x_1$

$|x_{k+1} - x_k| \leq \frac{L}{2^k}$

$|y_{k+1} - y_k| \leq \frac{L}{2^k}$

Thus, for $\ell > k$, $x_\ell - x_k = \sum_{j=k}^{\ell-1} x_{j+1} - x_j$

$$|x_\ell - x_k| \le \sum_{j=k}^{\ell-1} |x_{j+1} - x_j| \le \sum_{j=k}^{\ell-1} \frac{L}{2^j} \le \sum_{j=k}^{\infty} \frac{L}{2^j} = \frac{L}{2^{k-1}}$$

$$|y_\ell - y_k| \le \frac{L}{2^{k-1}}$$

$\Rightarrow \{x_k\}, \{y_k\}$ Cauchy with equal limit,

Let $\{x_k\} \to y$
$\{y_k\} \to y$

- y is an upper bound: Let $e \in E$. Then $e \le y_k$
$\Rightarrow e \le y$ (persistance of non-strict ineq's wrt limits)

- y is the least upper bound:
Let $y' < y$.
$\exists\, m$ s.t. $k \ge m \Rightarrow y' < x_k$, since $\{x_k\} \to y$.
x_k not u.b. $\Rightarrow \exists\, e \in E$ s.t. $y' < x_k < e \Rightarrow y'$ not u.b. ∎

Def: $\{x_k\}$ is monotone increasing iff $x_{k+1} \ge x_k \ \forall k$
" decreasing " $x_{k+1} \le x_k \ \forall k$

Theorem Let $\{x_k\} \subset \mathbb{R}$, monotone increasing and bounded above. Let $x = \sup\{x_k\}$
Then $\lim_{k\to\infty} x_k = x$. (i.e. monotone increasing seq's that are bd above converge)

proof: Let $n > 0$. $x - \frac{1}{n}$ is not an upper bound for $\{x_k\}$.
$\Rightarrow \exists\, m$ s.t. $x_m > x - \frac{1}{n}$
$\ell \ge m \Rightarrow x_\ell > x - \frac{1}{n}$. $x = \sup\{x_k\} \Rightarrow x - \frac{1}{n} < x_\ell \le x$
$\Rightarrow |x - x_\ell| < \frac{1}{n}$ ∎

(Also, if a monotone decreasing sequence is bounded below, it converges to its infimum)

Def Let $\{x_k\} \subset \mathbb{R}$ a sequence.
x is a limit point of $\{x_k\}$ iff
a) $\exists$ subsequence $\{y_j\} = \{x_{k_j}\}$ $k_{j+1} > k_j$ s.t. $\{y_j\} \to x$
or equivalently,
b) $\forall n \in \mathbb{N}$ $\exists$ ∞'ly many j s.t. $|x_j - x| < \frac{1}{n}$

Check equivalence:

Example : Consider $\{.1, .2, \ldots .9, .01, .02, \ldots .99, .001, .002, \ldots .999, \ldots\}$.
What are the limit points of this sequence?
proof!

Example Can you construct a sequence so that the set of limit points is all of $\mathbb{R}$?

Def : Let $\{x_k\}$ be a bounded sequence in $\mathbb{R}$

$$\limsup_{k\to\infty} \{x_k\} := \lim_{k\to\infty} \underbrace{\left(\sup_{j\geq k} x_j\right)}_{:y_k} \; ; \; \{y_k\} \searrow, \text{ bd below by } \inf\{x_k\}, \text{ so limit exists.}$$

$$\liminf_{k\to\infty} \{x_k\} := \lim_{k\to\infty} \left(\inf_{j\geq k} x_j\right)$$

in example above, what are liminf & limsup?

Theorem Let $\{x_k\}$ be a bounded sequence. Then

(a) $\limsup\limits_{k\to\infty} \{x_k\} := X$, $\liminf\limits_{k\to\infty} \{x_k\} := x$ are limit points of $\{x_k\}$

(b) every limit point z of $\{x_k\}$ satisfies $x \leq z \leq X$

(c) $\{x_k\}$ converges iff $x = X$

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proof: (a) Show X is a limit point; x analogous :

Let $X_k := \sup\limits_{j\geq k} x_j$. $\{X_k\} \searrow X$

pick k_1 s.t $x_{k_1} > X - 1$ (∞'ly many x_k satisfy this, since each $X_k \geq X$)

pick $k_2 > k_1$ s.t. $x_{k_2} > X - \frac{1}{2}$

$\vdots$

$k_j > k_{j-1}$ s.t. $x_{k_j} > X - \frac{1}{j}$

Thus
$$\begin{array}{ccc} X - \frac{1}{j} & < x_{k_j} \le & X_{k_j} \\ \big\downarrow {\scriptstyle j\to\infty} & & \big\downarrow {\scriptstyle j\to\infty} \\ X & & X \end{array}$$
because subsequences of convergent sequences converge to same limit (below)

So by sandwich lemma (below), $\{x_{k_j}\} \to X$ ∎

(b) Let $\{x_{k_j}\} \to z$.
$$\begin{array}{ccc} x_{k_j} & \le & X_{k_j} \\ \downarrow & & \downarrow \\ z & \le & X \end{array}$$

Similarly, $x \le z$.

(c) Let $\{x_k\} \to z$. Let $n > 0$. $\exists\, m$ s.t. $k \ge m \Rightarrow |z - x_k| < \frac{1}{n}$

$\Rightarrow z - \frac{1}{n} < x_k < z + \frac{1}{n}$

$\Rightarrow z - \frac{1}{n} \le \inf\limits_{k \ge m} \{x_k\} \le \sup\limits_{k \ge m} \{x_k\} \le z + \frac{1}{n}$

$\Rightarrow z - \frac{1}{n} \le x \le X \le z + \frac{1}{n}$

Let $n \to 0$; $\Rightarrow z \le x \le X \le z \Rightarrow x = z = X$.

Conversely, if
$x = X$

then
$$\begin{array}{ccccc} \inf\limits_{j \ge k} x_j & \le & x_k & \le & \sup\limits_{j \ge k} x_j \\ \downarrow & & & & \downarrow \\ x & & = & & X \end{array}$$

$\Rightarrow \{x_k\} \to x = X$ sandwich!

<u>Subsequence lemma</u>: Let $\{x_k\} \to x$. Let $\{x_{k_j}\}$ a subsequence $(k_{j+1} > k_j)$. Then $\{x_{k_j}\} \to x$.

pf: Let $n > 0$. $\exists\, m$ s.t. $k \ge m \Rightarrow |x - x_k| < \frac{1}{n}$. Pick m_2 s.t. $j \ge m_2 \Rightarrow k_j \ge m$. $\Rightarrow |x_{k_j} - x| < 1/n$ ∎

<u>Sandwich lemma</u>: Let $x_k \le y_k \le z_k \quad \forall k$.

If $\{x_k\} \to a$, $\{z_k\} \to a$, then also $\{y_k\} \to a$.

<u>proof</u>
$$\begin{aligned} |y_k - a| &= |y_k - x_k + x_k - z_k + z_k - a| \\ &\le \underbrace{|y_k - x_k|}_{\le |z_k - x_k| \,\to\, 0} + \underbrace{|x_k - z_k|}_{\to\, 0} + \underbrace{|z_k - a|}_{\to\, 0} \end{aligned}$$
∎