Math 5210
Tuesday Jan 20 §2.3

- page 4 Friday's notes,

  $\mathbb{Q}$ is <u>dense</u> in $\mathbb{R}$
  and
  $\mathbb{R}$ is <u>complete</u> (ordered Archimedian field)

- <u>Limit theorems</u> (in $\mathbb{R}$). Let $\{x_k\} \to x$ and $\{y_k\} \to y$. Then

  a) $\{x_k + y_k\} \to x + y$

  $\{x_k y_k\} \to xy$

  b) If $y \neq 0$ then $\exists\, m$ s.t. $y_k \neq 0\ \forall k \geq m$, and for $k \geq m$ $\{x_k/y_k\} \to x/y$.

  c) If $\exists\, m$ s.t. $x_k \geq y_k\ \forall k \geq m$, then $x \geq y$.

(a), (b): We did all necessary estimates for these when we were checking that $+, \cdot$ were well-defined for $\mathbb{R}$, although the context was different.
You check (a) for sums in HW (& should be able to do product & quotient estimates too!)

(c): Suppose on the contrary that $x < y$: $\Rightarrow \exists\, n$ s.t. $x - y < -\frac{1}{n} < 0$ (Archimedian ordering)

$\Rightarrow \exists\, m_1$ s.t. $k \geq m_1 \Rightarrow x_k - y_k < -\frac{1}{2n}$ $\Rightarrow\Leftarrow$ ∎

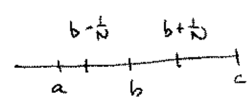
(implied by $|x - y - (x_k - y_k)| < \frac{1}{2n}$).

Another useful fact:

(d) Let $\{x_k\} \to b$, $a < b < c$.

Then $\exists\, m$ s.t. $k \geq m \Rightarrow a < x_k < c$

<u>proof</u>: by Archimedian property $\exists\, N$ s.t. $b - a > \frac{1}{N}$, $c - b > \frac{1}{N}$



$\exists\, m$ s.t. $k \geq m \Rightarrow |x_k - b| < 1/N$

$\Leftrightarrow -\frac{1}{N} < x_k - b < \frac{1}{N}$

$\Leftrightarrow a < b - \frac{1}{N} < x_k < b + \frac{1}{N} < c$ ∎

(You might use this sort of idea on 2.2.4 #5).

Decimal expansions (See also 2.4.1)

Def. The decimal expansion $N.a_1a_2a_3\cdots a_k\cdots$ $\quad N \in \mathbb{Z}$, $a_j \in \{0,1,..9\}$

represents the real number

$$x = [\{x_k\}] \quad \text{where } x_k = N + \sum_{j=1}^{k} a_j \cdot 10^{-j}$$

(We assume here $x \geq 0$, i.e. $N \geq 0$. $x<0 \Rightarrow x=-y$ with y representable as above.)

Theorem Every real number has a decimal expansion

proof: We assume $x \geq 0$.

By the Archimedian property and the axiom that any finite collection of integers has a largest el.,

$\exists\, N \in \{0,1,2,..\}$ s.t.

$$N \leq x < N+1 \qquad \text{(why?)}$$

Thus $0 \leq x-N < 1$.

$\exists!\, a_1$ s.t. $\frac{a_1}{10} \leq x-N < \frac{a_1+1}{10}$ $\quad a_1 \in \{0,1,..9\}$

$\exists!\, a_2$ s.t. $\frac{a_2}{10^2} \leq x-N-\frac{a_1}{10} < \frac{a_2+1}{10^2}$ $\quad a_2 \in \{0,1,..9\}$.

inductively, $\exists!\, a_k$ s.t. $a_k 10^{-k} \leq x - N - \sum_{j=1}^{k-1} a_j 10^{-j} < (a_k+1)\,10^{-k}$ $\quad a_k \in \{0,1,..9\}$.

Thus, the sequence $\{x_k\}$ with $x_k = N + \sum_{j=1}^{k} a_j 10^{-j}$

is Cauchy and converges to x, i.e. $[\{x_k\}] = x$.

Theorem $x \in \mathbb{R}$ is rational iff it has an eventually repeating decimal expansion (you did in HW!)

Theorem let x have decimal expansion $x = N.a_1a_2\cdots a_k\cdots$

If no power of 10 times x is an integer, this expansion is unique.

If $10^k x$ is an integer ($k \in \mathbb{Z}$) and $10^{k-1}x$ is not an integer, let $10^k x = N_1$.

then x has exactly two possible decimal expansions,

$$x = 10^{-k}(N_1.\bar{0}) \text{ and } 10^{-k}((N_1-1) + .9\bar{9}\ldots)$$

proof: If $N.a_1a_2\cdots = M.b_1b_2\cdots$

this means $N + \sum_{j=1}^{\infty} a_j 10^{-j} = M + \sum_{j=1}^{\infty} b_j 10^{-j}$

i.e. $(N-M) + \sum_{j=1}^{\infty} (a_j - b_j) 10^{-j} = 0$.

The result follows from the fact that $\sum_{j=1}^{\infty} c_j 10^{-j} < 1$ for $c_j \in \{0,1,2,..9\}$ if any $c_j \neq 9$

$= 1$ iff all $c_j = 9$

(check the details!)

Example: Show $\sqrt{2}$ exists in $\mathbb{R}$, i.e. $\exists\, x>0$ s.t. $x^2=2$. In fact, x is unique.

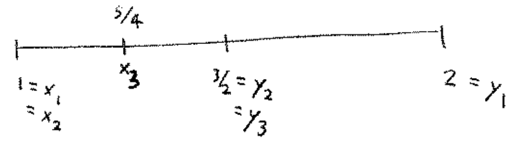
We use a subdivision argument (not numerically efficient, but easy to show Cauchyness of approximations)

Let $x_1 = 1$, $y_1 = 2$; $\quad x_1^2 < 2 < y_1^2$, $\quad y_1 - x_1 = 1$

Let $z = \frac{x_1+y_1}{2} = 3/2$

Since $z^2 > 2$ define

$x_2 = 1$, $y_2 = 3/2$ $\quad x_2^2 < 2 < y_2^2$, $\quad y_2 - x_2 = 1/2$



Inductively, given $x_k < y_k$ $\quad x_k^2 < 2 < y_k^2$, $\quad y_k - x_k = \frac{1}{2^{k-1}}$

let $z = \frac{x_k+y_k}{2}$. If $z^2<2$ define $x_{k+1} = z$, $y_{k+1} = y_k$

If $z^2 > 2$ define $x_{k+1} = x_k$, $y_{k+1} = z$

($z^2 = 2$ not possible, since $\sqrt{2}$ is irrational)

$|x_{k+1} - x_k| \le \frac{1}{2^k}$ so for $\ell > k$, $|x_\ell - x_k| \le \sum_{j=k}^{\ell-1} |x_{j+1} - x_j| < \sum_{j=k}^{\infty} \frac{1}{2^j} = \frac{1}{2^{k-1}}$

Thus $\{x_k\}$ and by identical estimate $\{y_k\}$ are Cauchy.

$\{x_k\} \equiv \{y_k\}$ since $|x_k - y_k| = \frac{1}{2^{k-1}} \to 0$ as $k \to \infty$.

Let $x := \lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k$

$x_k^2 < 2 \Rightarrow x^2 \le 2$
$y_k^2 > 2 \Rightarrow x^2 \ge 2$ $\quad$ Thus $x^2 = 2$.

x is unique since if $x, y > 0$ satisfy $x^2 = 2$ and $y^2 = 2$ then $0 = x^2 - y^2 = (x-y)\underbrace{(x+y)}_{>0}$, so $0 = x - y$.

Examples: It takes more machinery, but you could prove (assuming integral properties) that e exists: the unique $x>1$ s.t. $\int_1^x \frac{1}{t}\,dt = 1$ $\quad$ subdivision would work here too!

Can you think of a subdivision method which yields a Cauchy sequence for π?