Math 5210
Wed 1/14
Problem sessions start tomorrow Thurs.
2-3:15 LCB 222

Real arithmetic §2.2

Recall (for sequences of rational #'s)

$\{x_k\}$ Cauchy :=

if $\{x_k\}, \{y_k\}$ Cauchy, then $\{x_k\} \equiv \{y_k\}$ iff

the real #'s $\mathbb{R}$ are:*

* We need to show $\equiv$ is an equivalence relation!

Notation: if $\{x_k\}$ is a Cauchy sequence (today all sequences are of rational numbers)
we will write

$[\{x_k\}] \in \mathbb{R}$

the equivalence class of $\{x_k\}$.

Remark: if we identify the rational number $q \in \mathbb{Q}$
with the Cauchy sequence $\{q, q, q \cdots\}$
and with $[\{q, q, q, \ldots\}] \in \mathbb{R}$

we can think of $\mathbb{Q} \subset \mathbb{R}$:

This is because

$i(q) := [\{q, q, \ldots\}]$ is injective (1-1)

pf: if $p \neq q$ then $|p-q| > 0$
so $\exists\, n \in \mathbb{N}$ with $|p-q| > \frac{1}{n}$ (Arch. lemma)

therefore $\{p, p, p \cdots\} \not\equiv \{q, q, q, \ldots\}$

so $[\{p, p, p \ldots\}] \neq [\{q, q, q\}]$
$i(p) \neq i(q)$.

We aim to verify today that
real arithmetic is well-defined, and
extends rational arithmetic. Of course, you've always taken this for granted!

<u>Def 1</u>: $[\{x_k\}] + [\{y_k\}] := [\{x_k + y_k\}]$

$$[\{x_k\}] \cdot [\{y_k\}] := [\{x_k y_k\}]$$

$$-[\{x_k\}] := [\{-x_k\}]$$

Of course, we must check that these definitions make sense – (i) the sequences on the right must be Cauchy

(ii) the definitions cannot depend on equivalence class representatives

The following lemmas will make our life easier to check "well-defined" above, and subsequently

<u>Lemma 1</u> (Cauchy $\Rightarrow$ bounded): $\{x_k\}$ Cauchy $\Rightarrow \exists M$ s.t. $|x_k| \le M \ \forall k$.

pf: for $n=1$ pick m s.t. $|x_j - x_k| < 1/n = 1 \quad \forall j,k \ge m$

$\Rightarrow |x_j - x_m| < 1 \quad \forall j \ge m$.

$\Rightarrow |x_j| = |x_j - x_m + x_m|$

$< 1 + |x_m| \quad \forall j \ge m$

Define $M = \max\{|x_1|, |x_2|, \dots |x_{m-1}|, |x_m| + 1\}$. ∎

<u>Lemma 2</u> (Easier Cauchy and Limits)

2a) Let $\{x_k\}$ be a sequence. If $\exists\, C>0$ s.t. $\forall n>0\ \exists\, m$ s.t. $j,k \ge m \Rightarrow |x_j - x_k| < \frac{C}{n}$ then $\{x_k\}$ is Cauchy

2b) To prove $\{x_k\} \to L$ it suffices to find $C>0$ s.t. $\forall n>0\ \exists\, m$ s.t. $k \ge m \Rightarrow |x_k - L| < \frac{C}{n}$

2c) If $\{x_k\}, \{y_k\}$ are Cauchy and $\exists\, C>0$ s.t. $\forall n>0\ \exists\, m>0$ s.t. $k \ge m \Rightarrow |x_k - y_k| < C/n$, then $\{x_k\} \equiv \{y_k\}$

proofs:

2a) Assume hypothesis, with given C. $\boxed{\text{Let } n'>0.}$ Pick n s.t. $\frac{C}{n} < \frac{1}{n'}$ (iff $Cn' < n$)

For this n $\exists\, m$ s.t. $\boxed{j,k \ge m} \Rightarrow |x_j - x_k| < \frac{C}{n}$

thus $\boxed{|x_j - x_k| < \frac{1}{n'}}$ ∎

2b) (Similar to 2a): If C exists, let $n'>0$, then pick n with $\frac{C}{n} < \frac{1}{n'}$. Then $\exists\, m$ s.t. $k \ge m$ $\Rightarrow |x_k - L| < \frac{C}{n} < \frac{1}{n'}$

2c) $\{x_k\} \equiv \{y_k\}$ iff $\{x_k - y_k\} \to 0$.

By (2b), $\{x_k - y_k\} \to 0$ if $\exists\, C>0$ s.t. $\forall n>0\ \exists\, m>0$ s.t. $k \ge m \Rightarrow |x_k - y_k - 0| < \frac{C}{n}$; 2c! ∎

Theorem 1: Definitions 1 are well-defined: Let $\{x_k\}, \{x_k'\}, \{y_k\}, \{y_k'\}$ Cauchy
Let $\{x_k\} \equiv \{x_k'\}$, $\{y_k\} \equiv \{y_k'\}$

Then (i) $\{x_k+y_k\}$, $\{x_k y_k\}$, $\{-x_k\}$ are Cauchy (also with $'$s)

(ii) $\{x_k+y_k\} \equiv \{x_k'+y_k'\}$

$\{x_k y_k\} \equiv \{x_k' y_k'\}$

$\{-x_k\} \equiv \{-x_k'\}$

Proof: (i) $\{-x_k\}$ is clearly Cauchy.

sum: Let $n>0$. $|x_j+y_j-(x_k+y_k)| \leq |x_j-x_k| + |y_j-y_k|$

pick m_1 s.t. $j,k \geq m_1 \Rightarrow |x_j-x_k| < \frac{1}{n}$

m_2 s.t. $j,k \geq m_2 \Rightarrow |y_j-y_k| < \frac{1}{n}$

So, for $m = \max(m_1,m_2)$

$$|(x_j+y_j)-(x_k+y_k)| < \frac{2}{n} \quad \blacksquare \quad \text{(Lemma 2)}.$$

product: Let $n>0$. Pick m_1, m_2 as above, also m.

Let M_1 s.t. $|x_k| \leq M_1 \ \forall k$

M_2 s.t. $|y_k| \leq M_2 \ \forall k$

then for $j,k \geq m$:
$$|x_j y_j - x_k y_k| = |x_j(y_j-y_k) + y_k(x_j-x_k)|$$
$$\leq M_1\left(\frac{1}{n}\right) + M_2\left(\frac{1}{n}\right) = \frac{C}{n}$$

$C := M_1+M_2$

so $\{x_k y_k\}$ is Cauchy (Lemma 2)

$\blacksquare$

(ii) $\{-x_k\} \equiv \{-x_k'\}$ clear

Let $n>0$. $\exists$ m_1 s.t. $k \geq m_1 \Rightarrow |x_k-x_k'| < \frac{1}{n}$

m_2 s.t. $k \geq m_2 \Rightarrow |y_k-y_k'| < 1/n$

for $k \geq m = \max(m_1,m_2)$

sum $\quad |(x_k+y_k)-(x_k'+y_k')| \leq \frac{2}{n} \Rightarrow \{x_k+y_k\} \equiv \{x_k'+y_k'\}$.

product $\quad |x_k y_k - x_k' y_k'| = |x_k(y_k-y_k') + y_k'(x_k-x_k')|$

$$\leq \frac{M_1}{n} + \frac{M_2}{n} = \frac{C}{n}$$

where $M_1 := \max_k |x_k|$

$M_2 := \max_k |y_k'|$.

$\Rightarrow \{x_k y_k\} \equiv \{x_k' y_k'\} \quad \blacksquare$

Theorem 2 The field axioms hold for $\mathbb{R}$

1. $a+b=b+a \qquad (a+b)+c = a+(b+c)$
   $a\cdot b = b\cdot a \qquad (a\cdot b)c = a\cdot(b\cdot c)$
2. $a\cdot(b+c) = ab+ac$
3. $0+a=a$
   $1\cdot a = a$
4. $a+(-a)=0$
   $a\cdot\frac{1}{a} = 1 \quad a\neq 0.$

pf: 1-3 are straightforward, as is additive inverse in 4, using Theorem 1

e.g. $(a\cdot b)c = a\cdot(b\cdot c) \longrightarrow [\{x_k\}]([\{y_k\}][\{z_k\}])$

$$\underbrace{([\{x_k\}]\cdot[\{y_k\}])}\cdot[\{z_k\}] \qquad [\{x_k\}]\cdot\underbrace{[\{y_k\}][\{z_k\}]}$$

$$\underbrace{[\{x_k y_k\}]\cdot[\{z_k\}]} \qquad \underbrace{[\{x_k\}]\cdot[\{y_k z_k\}]}$$

$$[\{(x_k y_k) z_k\}] = [\{x_k y_k z_k\}]$$

in (3): $0 := [\{0,0,\ldots\}]$
$1 := [\{1,1,\ldots\}]$

We need to be a little careful in 4b: multiplicative inverses ....
we'll return to this after discussing ordering.

Theorem 3 Let $x = [\{x_k\}] \in \mathbb{R}$. Then exactly one of the following holds:

a) $x=0$

b) $\exists\, n>0$ and $m>0$ s.t. $\forall k\geq m \quad x_k > \frac{1}{n}$. In this case we say $x>0$
and for each $\{y_k\}\equiv\{x_k\}$ $\exists\, n', m'>0$ s.t. $\forall k\geq m'$, $y_k > 1/n'$

or c) $\exists\, n>0$ and $m>0$ s.t. $\forall k\geq m \quad x_k < -\frac{1}{n}$ In this case we say $x<0$
and for each $\{y_k\}\equiv\{x_k\}$ $\exists\, n', m'>0$ s.t.
$\forall k\geq m'$, $y_k < -\frac{1}{n'}$.

This defines an ordering on $\mathbb{R}$

proof of Theorem 3

- Cases (b), (c) are mutually exclusive.
- If (b) holds and $\{y_k\} \equiv \{x_k\}$, pick $n, m > 0$ s.t. $x_k > \frac{1}{n}$ $\forall k \geq m$.

For $n_1 = 2n$ pick m_1 s.t. $|x_k - y_k| \leq \frac{1}{2n}$ $\forall k \geq m_1$

Let $n' = 2n$, $m' = \max(m, m_1)$, then $k \geq m' \Rightarrow$

$$y_k = y_k - x_k + x_k \geq -\frac{1}{2n} + \frac{1}{n} = \frac{1}{2n} = \frac{1}{n'}$$

∎

- If (c) holds then the analogous statement to the one in (b), holds for $\{y_k\} \equiv \{x_k\}$

- If neither (b) nor (c) hold, we shall show $x = 0$.

If (b) doesn't hold, it is not true that

$\exists n > 0$ for which $\exists m > 0$ s.t. $\forall k \geq m$, $x_k > \frac{1}{n}$

i.e. $\nexists n > 0$ s.t. $\exists m > 0$ s.t. $\forall k \geq m$, $x_k > \frac{1}{n}$

i.e. $\forall n > 0$ $\nexists m > 0$ s.t. $\forall k \geq m$, $x_k > \frac{1}{n}$

i.e. $\forall n > 0$ and $\forall m > 0$ $\exists k \geq m$ s.t. $x_k \leq 1/n$

But $\{x_k\}$ is Cauchy, so

$\forall n > 0$ $\exists m > 0$ s.t. $\forall j, l \geq m$, $|x_j - x_l| < \frac{1}{n}$

for this m, find $k \geq m$ s.t. $x_k \leq \frac{1}{n}$

then $j \geq m \Rightarrow |x_j - x_k| \leq 1/n$

$\Rightarrow x_j = x_k + (x_j - x_k)$

so $x_j \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$

Thus $\forall n > 0$ $\exists m > 0$ s.t.

$j \geq m \Rightarrow x_j \leq 2/n$

If (c) doesn't hold, deduce

$\forall n > 0$ $\exists \tilde{m} > 0$ s.t.

$j \geq \tilde{m} \Rightarrow x_j \geq -\frac{2}{n}$

Let $n > 0$. Let $\tilde{\tilde{m}} = \max(m, \tilde{m})$.

Then $j \geq \tilde{\tilde{m}}$

$\Rightarrow |x_j| \leq \frac{2}{n}$

Thus $[\{x_j\}] = 0$.

∎

(Still need to check mult inverses, prove Δ ineq., etc.)