Math 5210
Monday Feb. 9   9.3 cont'd.

- Recall the 3 equiv def'ns of $f: X \to Y$ continuous
- Finish p. 3 Friday, which reminds you of how the zoo of continuous functions is built
- Example p. 4 Friday

Then, theorems about continuous functions and compact sets: (proofs you will be responsible for.)

<u>Theorem 1</u> Let $f: (X, d_1) \to (Y, d_2)$ continuous, and let X compact
Then the image $f(X) = \{y \in Y \text{ s.t } y = f(x) \text{ for some } x \in X\}$ is compact

pf 1: use sequential compactness:

pf 2: use H.B. property:

<u>Corollary</u>: Let $f: (X, d) \to \mathbb{R}$ continuous, X compact
Then f attains its suprema,
i.e. $\exists$ $x_1 \in X$ s.t. $f(x_1) = \sup_{x \in X} f(x)$
$x_2 \in X$ s.t. $f(x_2) = \inf_{x \in X} f(x)$

$\Rightarrow$ Extreme value theorem in Calculus

pf:

Theorem 2 Let $f:(X,d_1)\to(Y,d_2)$ continuous, X compact.

Then f is uniformly continuous

$\downarrow$

pf: $\forall\, \varepsilon>0\ \exists\, \delta>0$ s.t. $\forall\, x,z$ s.t. $d(x,z)<\delta$,

$d(f(x),f(z))<\varepsilon$.

for $f:[a,b]\to\mathbb{R}$ this theorem was crucial for defining the Riemann integral in 3210.

Theorem 3 Let X compact, Y complete.

Define $C(X,Y)=\{f: X\to Y \text{ s.t. } f \text{ continuous}\}$

For $f,g\in C(X,Y)$ define $d(f,g)=\sup\limits_{x\in X} d(f(x),g(x)) = \max\limits_{x\in X} d(f(x),g(x))$

Then d is a distance on $C(X,Y)$,

and $(C(X,Y),d)$ is complete.

pf: (mirrors the proof that $(C([a,b]),\|\cdot\|_\infty)$ is complete, jan27.pdf)

We'll return to (and prove) this last theorem later ~ you essentially proved one direction of it in last weeks' hw:

<u>Theorem 4</u> (Arzela-Ascoli). Let X and Y be compact.
Let $A \subset C(X,Y)$

Then A is compact iff A is closed and <u>uniformly equicontinuous</u>

$$\uparrow$$

$\forall n \in \mathbb{N} \ \exists \ m \in \mathbb{N}$ s.t.
$\forall f \in A$,
$d(x,z) < \frac{1}{m}$
$\Rightarrow d(f(x), f(z)) < \frac{1}{n}$.