Math 5210
Friday Feb 6
- Finish HB $\Rightarrow$ sequential compactness (Wed)

HW for Fri 2/13 ①
- carry over this week's $^\circ, ^-, ^c$ exercise
Also 9.3 1, 2, 3, 6, 10, 7, 15 (and show that if you use the closed intervals $[0,1]$, then you do have a fixed point) 16, 20, 26.

§9.3 Continuity

Let $f : (M, d_1) \to (N, d_2)$. be a function
$\uparrow$ domain $\uparrow$ range

for $A \subset M$, $f(A) := \{ f(x) \text{ s.t. } x \in A \}$ is the <u>image</u> of A
for $B \subset N$, $f^{-1}(B) := \{ x \in A \text{ s.t. } f(x) \in B \}$ is the <u>preimage</u> of B.

<u>Theorem 1</u> Let f as above. The following 3 conditions are equivalent.
If one (hence all) of them holds, we say that f is <u>continuous</u>

1) $\forall x_0 \in M, n \in \mathbb{N} \ \exists m \in \mathbb{N}$ s.t. $d(x, x_0) < \frac{1}{m} \Rightarrow d(f(x), f(x_0)) < \frac{1}{n}$

(if this holds at x_0, "f is cont. at x_0."
So (1) is saying f is cont at every x_0)

(you could also say
$\forall x_0 \in M, \varepsilon > 0, \exists \delta > 0$ s.t. $d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$)

2) $\forall x_0 \in M$, $\{x_k\} \to x_0$ then $\{f(x_k)\} \to f(x_0)$
(if this holds at x_0, "f is sequentially cont. at x_0"
So (2) is saying f is sequentially cont at all x_0)

3) $\forall \mathcal{O} \subset N$ open, $f^{-1}(\mathcal{O}) \subset M$ is open
("inverse images of open sets are open")

This Theorem is useful because different situations lend themselves to these different ways of describing "continuous".

Proof:

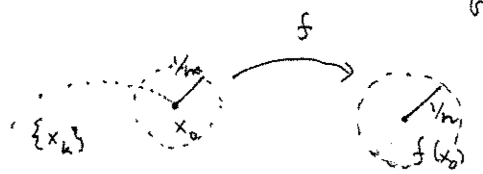
There are various paths which you can use to show the equivalence of these 3 definitions - it's fun to pick any two of the conditions and then showing equivalence without using the 3rd. But,

we'll show ① $\Rightarrow$ ② $\Rightarrow$ ③ $\Rightarrow$ ①.

① $\Rightarrow$ ②. Let $\{x_k\} \to x_0$. f is cont at x_0, so for $n \in \mathbb{N}$ $\exists$ m s.t. $d(x, x_0) < \frac{1}{m} \Rightarrow d(f(x), f(x_0)) < \frac{1}{n}$.

but $\{x_k\} \to x_0$ so $\exists$ m_1 s.t. $k \geq m_1 \Rightarrow d(x_k, x_0) < \frac{1}{m}$

$\Rightarrow d(f(x), f(x_0)) < \frac{1}{n}$ ∎



② $\Rightarrow$ ③ : To show $\mathcal{O} \subset N$ open $\Rightarrow f^{-1}(\mathcal{O})$ open in M

it is equivalent to show $C \subset N$ closed $\Rightarrow f^{-1}(C)$ closed in M (see HW!)

So, let $C \subset N$ closed.

Let $\{x_k\}$ a convergent sequence in $f^{-1}(C)$, with limit $x_0 \in M$

we must show $x_0 \in f^{-1}(C)$ as well (def of closed!)

But $\{x_k\} \to x_0 \Rightarrow \{f(x_k)\} \to f(x_0)$ by ②

since $\{f(x_k)\} \subset C$, so is the limit $f(x_0)$ (C closed!)

i.e. $x_0 \in f^{-1}(C)$ ∎

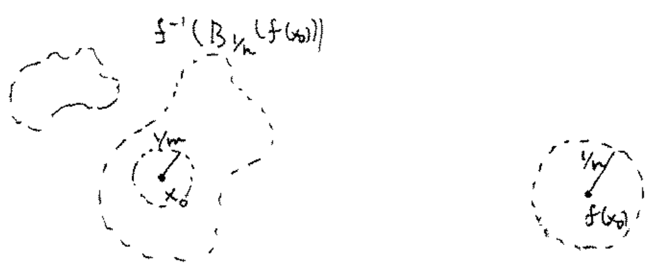
③ $\Rightarrow$ ① Let $n \in \mathbb{N}$, $x_0 \in M$. $B_{\frac{1}{n}}(f(x_0)) \subset N$ is open.

Thus $f^{-1}(B_{\frac{1}{n}}(f(x_0)))$ is too. (③)

$\ni x_0$, so $\exists$ m s.t. $B_{\frac{1}{m}}(x_0) \subset f^{-1}(B_{\frac{1}{n}}(f(x_0)))$

i.e. $d(x, x_0) < \frac{1}{m} \Rightarrow d(f(x), f(x_0)) < \frac{1}{n}$

∎

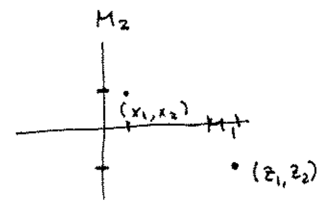


The zoo of continuous fcns.
(most of these are old friends.)

- constant maps $f: M \to N$, $f(x) = c \ \forall x \in M$.

- distance to a point, i.e. $f(x) = d(x, x_0)$ (See hw!)
  $f: M \to \mathbb{R}$

- More generally, let $(M_1, d_1), (M_2, d_2)$ be metric spaces.
  Then a natural metric to put on the product space
  $$M_1 \times M_2 := \{ (x_1, x_2) \text{ s.t. } x_1 \in M_1, x_2 \in M_2 \}$$
  is $$d((x_1,x_2),(z_1,z_2)) = \max\{ d(x_1,z_1), d(x_2,z_2)\}$$



  (If we did this for $\mathbb{R} \times \mathbb{R}$ this would be the $p = \infty$ metric)
  (we could also have used other p metrics, but they would all be equivalent to this one.)

  Then $d: M \times M \to \mathbb{R}$ is cont:
  Could you show that if $\{(x_k, y_k)\} \to (x,y)$ in the product distance then $\{d(x_k, y_k)\} \to d(x,y)$?

- If $f: (M,d) \to N$ is continuous
  and $A \subset M$, then $f(A,d) \to N$ is continuous.

- $f: \mathbb{R}^n \to \mathbb{R}$, $f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = x_i$ any $1 \le i \le n$ coord fcns.
  (because $\{\vec{x}_k\} \to \vec{x} \in \mathbb{R}^n$ iff each coord $\{x_{k,i}\}_k \to x_i$)

- If $\begin{matrix} f: M \to \mathbb{R} \\ g: M \to \mathbb{R} \end{matrix}$ are cont, then so are $f+g$, fg, and f/g (provided $g \neq 0$ on M)
  (use sequence characterization and Chapter 2 work! ∎)

- $f: M \to N_1 \times N_2$ is continuous iff each component function is
  $f(x) = (f_1(x), f_2(x))$
  pf: If $\{x_k\} \to x$ then
  $$d(f(x_k), f(x)) = \max\{ d_1(f_1(x_k), f_1(x)), d_2(f_2(x_k), f_2(x))\}$$
  so $\{f(x_k)\} \to f(x)$ iff $\{f_1(x_k)\} \to f_1(x)$ and $\{f_2(x_k)\} \to f_2(x)$ ∎

- (e.g.) $f: M \to \mathbb{R}^n$ is cont iff each component fcn is

- $\begin{matrix} f: M \to N \text{ cont} \\ g: N \to P \text{ cont} \end{matrix} \Rightarrow g \circ f: M \to P$ is (hw)

And so is built the zoo of cont. fcns...

An interesting example...

On page 3, we showed

$f: M \to N_1 \times N_2$ is cont iff each component function f_i is
$f(x) = (f_1(x), f_2(x))$

So it would be tempting to conjecture that

$f: M_1 \times M_2 \to N$ is cont iff $\forall x_0 \in M_1$, $f(x_0, y): M_2 \to N$ is cont
$y \mapsto f(x_0, y)$
and $\forall y_0 \in M_2$, $f(x, y_0): M_1 \to N$ is cont.

$\Rightarrow$ is true, since by fixing a coord you are restricting a cont. fcn on the space $M_1 \times M_2$ to a subspace.

but, $\Leftarrow$ is false in general:

e.g. $f: \mathbb{R}^2 \to \mathbb{R}$.

let θ, $0 \le \theta \le 2\pi$ be the usual polar coord fcn on $\mathbb{R}^2 \setminus \{0\}$ (which is continuous, except for the issue $\theta \equiv 2\pi$)
let $g(\theta)$, $g: [0, 2\pi] \to \mathbb{R}$ cont, s.t. g is not constant
$g(0) = g(2\pi)$
$g(0) = g(\pi) = g(\pi/2) = g(\frac{3}{2}\pi) = \beta$

then $f(x,y) = \begin{cases} g \circ \theta(x,y) & (x,y) \neq (0,0) \\ \beta & (x,y) = (0,0) \end{cases}$

is cont along each coord line
constant along each ray from the origin ($\setminus \{(0,0)\}$)
discontinuous at $(0,0)$.

e.g. $f(x,y) = \sin 2\theta = 2 \sin\theta \cos\theta$
$= 2 \frac{yx}{(x^2+y^2)}$