Math 5210
Tuesday Feb. 3 : Compactness notes

<u>Remark</u> : Strichartz' definition of limit points for sets $A$ is non-standard (see below), and so also is his definition of $\bar{A}$ (although the $\bar{A}$ def is equivalent to the usual one.)
I think he made this definition so that for a set $\{a_k\}_{k\in\mathbb{N}}$, any limit point of the set is actually a limit of a subsequence $\{a_{k_j}\}$: Let $x$ s.t. $\forall j \ \exists\ a_{k_j}\neq x$, $d(a_{k_j},x)<\frac{1}{j}$.
Then $\{a_{k_j}\}\to\infty$, so after rearranging so that $k_{j+1}\geq k_j$, and then discarding terms when $k_{j+1}=k_j$, we get a subsequence $\{a_{k_j}\}\to x$. This helps explain a technical point in a key compactness thm - page 5 today.

The usual definition of $\bar{A}$ is below:

<u>Lemma</u> Let $A\subset(X,d)$

a) Define $B=\{x\in X \text{ s.t. } \exists\ \{a_k\}\subset A \text{ s.t. } \{a_k\}\to x\}$. (Other authors would call $B$ the limit points of $A$)
Then $B=\bar{A}$

b) $A$ is closed iff $\{a_k\}\subset A$, $\{a_k\}\to x \Rightarrow x\in A$, i.e. iff $A=\bar{A}$

notice that (b) follows from (a), since the (old) def of $A$ closed, i.e. that it contain its limit points, is equivalent to $\bar{A}=A$, and by (a), $\bar{A}=B$.

proof of a).

$\bar{A}\subset B$ : $\bar{A}=A\cup\{\text{limit pts of } A\}$.
$A\subset B$, since $\{a,a,a,\ldots\}\to a$
$\{\text{limit pts of } A\}\subset B$ since $\forall k\in\mathbb{N}\ \exists\ a_k\in A$ $(a_k\neq x)$, $d(x,a_k)<1/k$, i.e. $\{a_k\}\to x$
thus $\bar{A}\subset B$.

$B\subset\bar{A}$ : $\bar{A}=A\cup\{\text{limit pts of } A \text{ not in } A\}$ (disjoint union)
if $x\in B$ and $x\notin A$, then $\exists\ \{a_k\}\subset A$, $a_k\neq x$, $\{a_k\}\to x$, so $x$ is a limit pt of $A$!
thus $B\subset\bar{A}$

Now begin the study of compactness:

<u>Definition</u> $(A,d)$ is compact iff $\forall\ \{a_k\}\subset A\ \exists$ subseq. $\{a_{k_j}\}\to a\in A$

Often $(A,d)\subset(X,d)$.
One case you studied in 3220 was $A\subset\mathbb{R}^n$.
You proved that <u>in $\mathbb{R}^n$</u>, $A$ is compact iff it is a closed and bounded subset of $\mathbb{R}^n$

- this theorem is reviewed in yesterday's notes, we'll discuss it.

**Remark** Closed and bounded subsets of metric spaces are NOT compact in general. For example, consider

$$\overline{B(0,1)} = \{ f \in C([0,1]) \text{ s.t. } \|f\|_{sup} \} \leq 1.$$

This set is closed and $\underbrace{\text{bounded}}_{d(f,0) \leq 1 \ \forall f \in \overline{B(0,1)}.}$

But not compact! In your HW you will find $\{f_k\} \subset \overline{B(0,1)}$

$\|f_k - f_j\|_{sup} = 1 \quad \forall k \neq j$

So no Convergent subsequence.

But there is a true theorem characterizing $(A,d)$ compact, which does generalize the $\mathbb{R}^n$ result:

**Def** $(A,d)$ is totally bounded iff $\forall r>0 \ \exists$ finitely many balls $\{B_r(a_j)\}_{j=1}^{N_r}$ s.t. $A \subset \bigcup_{j=1}^{n} B_r(a_j)$

**Theorem 1** $(A,d)$ is compact iff it is complete and totally bounded.

(closed subsets of $\mathbb{R}^n$ $\Leftrightarrow$ complete)

(bounded subsets of $\mathbb{R}^n$ $\Leftrightarrow$ totally bounded)

$\overline{B(0,1)} \subset (C([0,1], \|f\|_{sup})$ is not totally bounded.

**proof:** $\Rightarrow$: let $(A,d)$ compact.

let $\{a_k\} \subset A$ Cauchy.

Thus $\exists \ \{a_{k_j}\} \to a \in A$

$\Rightarrow$ the full sequence $\{a_k\} \to a$

$( d(a_k, a) \leq d(a_k, a_{k_j}) + d(a_{k_j}, a) )$

Thus $A$ is complete.

Next show $A$ is totally bounded:

Pf: let $r>0$. Pick $a_1 \in A$. If $A \subset B_r(a_1)$ you are done.

Else pick $a_2 \in A \setminus B_r(a_1)$

If $A \subset \bigcup_{j=1}^{2} B_r(a_j)$ you are done.

Continue inductively, in case $A \not\subset \bigcup_{j=1}^{k} B_r(a_j)$

pick $a_{k+1} \notin \bigcup_{j=1}^{k} B_r(a_j)$

If the process terminates, you've verified totally bounded for that r.

If not, your sequence of centers $\{a_k\}$ satisfies

$d(a_k, a_j) \geq r \quad \forall j,k$

$\Rightarrow$ no Cauchy subseq

$\Rightarrow$ no Conv. subseq

$\Rightarrow$ A not compact.

Thus, since A is compact, process terminates $\Rightarrow$ ■

$\Leftarrow$: Let $A$ be complete and totally bounded.

Let $\{a_k\} \subset A$.

Since $A$ can be covered by finitely many $r=1$ balls, $\exists$ one such ball $B_1(z_1)$ containing a subsequence of $\{a_k\}$

Let $a_{k_1}$ be in this subseq.

Similarly $\exists\ B_{1/2}(z_2)$ containing subseq. of $B_1(z_1)$ subseq.

Let $a_{k_2}$ $(k_2 > k_1)$ in this subsequence.

Induct.

Create subseq. $\{a_{k_j}\}$ s.t. $\ell \geq j \Rightarrow d(a_{k_\ell}, a_{k_j}) < \frac{1}{j}$

$\Rightarrow$ Cauchy

$\Rightarrow \{a_{k_j}\} \to a \in A$ since $A$ is complete

$\blacksquare$

The following (Heine-Borel) characterization of compact will be very useful:

<u>Theorem 2</u> $(A, d)$ is compact iff for every collection of open sets $\{\mathcal{O}_\beta\}_{\beta \in B}$ s.t. $\bigcup_{\beta \in B} \mathcal{O}_\beta \supset A$, $\exists$ a finite ~~subset~~ subcollection $\{\mathcal{O}_{\beta_i}\}_{i=1}^N$ s.t. $\bigcup_{i=1}^N \mathcal{O}_{\beta_i} \supset A$

"every open cover of $A$ has a finite subcover"

proof: $\Rightarrow$

Let $\{\mathcal{O}_\beta\}_{\beta \in B}$ an open cover of $A$

<u>Case I</u> ctble cover, $\{\mathcal{O}_k\}_{k \in \mathbb{N}}$

If $A \not\subset \mathcal{O}_1$ pick $a_1 \in \mathcal{O}_1^c$

If $A \not\subset (\mathcal{O}_1 \cup \mathcal{O}_2)$ pick $a_2 \in (\mathcal{O}_1 \cup \mathcal{O}_2)^c$

$\vdots$

If $A \not\subset \bigcup_{j=1}^k \mathcal{O}_j$, pick $a_k \in \left(\bigcup_{j=1}^k \mathcal{O}_j\right)^c$

If process terminates (i.e. $A \subset \bigcup_{j=1}^k \mathcal{O}_j$ for some $k$), we have a finite subcover

Otherwise get seq. $\{a_j\}$.

$A$ compact $\Rightarrow \{a_{j_k}\} \to a \in A$

$a \in \mathcal{O}_\ell$ some $\ell$ since original collect. covers.

$\Rightarrow a \in B_r(a) \subset \mathcal{O}_\ell$ some $r > 0$

$\Rightarrow \exists\ m$ s.t. $k \geq m \Rightarrow a_{j_k} \in \mathcal{O}_\ell$

but $j_k \geq \ell \Rightarrow a_{j_k} \notin \mathcal{O}_\ell$ $\Rightarrow\!\Leftarrow$

Thus process terminates, and $\exists$ finite subcover

<u>Case 2</u> $\{\mathcal{O}_\beta\}_{\beta \in B}$, $B$ not ctble.

In this case we use <u>Lemma 1</u>: $(A, d)$ compact implies $A$ is <u>separable</u>, i.e. $\exists\ \{z_k\} \subset A$ s.t. $\forall n \in \mathbb{N}, a \in A\ \exists\ z_k$ s.t. $d(a, z_k) < \frac{1}{n}$.

In other words, $\{z_k\}$ is a countable, <u>dense</u> subset of $A$.

proof: Since $A$ is totally bounded, $\forall n \in \mathbb{N}\ \exists\ \{z_k^n\}_{k=1}^{N_n}$ s.t.

$$A \subset \bigcup_{k=1}^{N_n} B_{\frac{1}{n}}(z_k^n)$$

The collection of centers $\{z_k^n\}_{\substack{n=1,2,\dots \\ k=1..N_n}}$ is a countable dense subset. ∎

<u>Lemma 2</u> $\mathcal{O} \subset A$ open

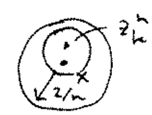
$$\Rightarrow \mathcal{O} = \bigcup_{\substack{n,k \text{ s.t.} \\ B_{\frac{1}{n}}(z_k^n) \subset \mathcal{O}}} B_{\frac{1}{n}}(z_k^n), \quad \text{where } B_{\frac{1}{n}}(z_k^n) \text{ as above.}$$

pf $\supset$: since each $B_{\frac{1}{n}}(z_k^n) \subset \mathcal{O}$.

$\subset$: Let $x \in \mathcal{O}$.

Let $B_{\frac{2}{n}}(x) \subset \mathcal{O}$. Pick $z_k^n$ s.t. $d(x, z_k^n) < \frac{1}{n}$

Then $x \in B_{\frac{1}{n}}(z_k^n) \subset B_{\frac{2}{n}}(x) \subset \mathcal{O}$. ∎




<u>Back to Case 2!</u>

Replace $\{\mathcal{O}_\beta\}_{\beta \in B}$ with

the countable cover $\{B_{\frac{1}{n}}(z_k^n) \text{ s.t. } B_{\frac{1}{n}}(z_k^n) \subset \mathcal{O}_\beta, \text{ for some } \beta\}$

This countable collection covers each $\mathcal{O}_\beta$, so also $A$.

By case 1 $\exists$ finite subcover

$$\begin{array}{ccc} B_{\frac{1}{n_1}}(z_{k_1}^{n_1}) & \cup \cdots \cup & B_{\frac{1}{n_\ell}}(z_{k_\ell}^{n_\ell}) \\ \cap & & \cap \\ \mathcal{O}_{\beta_1} & \cup \cdots \cup & \mathcal{O}_{\beta_\ell} \end{array}$$

∎ $\Longrightarrow$

$\Leftarrow$ : To show open cover property (HB) $\Rightarrow$ Compact,
we show not compact $\Rightarrow$ not HB:

Let $\{a_k\}$ s.t. $\{a_k\}$ has no convergent subsequence in $(A,d)$.
This implies $\{a_k\}$ is closed in A!!

(because any limit point would be a subsequence limit – page 1).

Similarly, $\{a_k\}_{k \geq m}$ is closed in $(A,d)$

Thus $\mathcal{O}_m = \{a_k\}_{k\geq m}^{C}$ is open.

$$\bigcup_m \mathcal{O}_m \supset A \quad \text{since any } a_j \in \mathcal{O}_m \text{ for } m > j.$$

By construction $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \cdots$ is increasing,
so if $\exists$ finite subcover of $\{\mathcal{O}_m\}$, deduce

$$A \subset \mathcal{O}_m \text{ for some } m.$$

$\nRightarrow$ since $a_m \notin \mathcal{O}_m$.

Thus H.B. does not hold!

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