HW for Friday March 6

7.3.4 2, 5, 7, 12
7.4.5 4, 7
7.5.5 4, 8, 10, 15

Math 5210

Friday Feb 27: §7.3, with applications to 11

<u>Where we are</u>: We've mostly finished chapter 11, using Riemann integral ideas from chapter 6, differentiability ideas from chapter 10, and metric space theory from chapter 9.

We're about to do chapter 7 topics which were not on the 3220 syllabus. (7.3, 7.5, 7.6) We'll relate some of these topics back to Chapters 9, 11, and use them in Chapter 12 (Fourier Series.)…. eventually we'll progress to the Lebesgue integral, chapter 14.

§7.3.2 Integration and differentiation of limits.

Let $f : [a,b] \to X_{\text{Banach}}$

Recall f is Riemann integrable on $[a,b]$ iff $\lim_{\|P\| \to 0} \sum_i f(x_i^*)\Delta x_i$ exists. (and the limit is called $\int_a^b f(x)dx$)

We proved f cont on $[a,b] \Rightarrow \int_a^b f(x)dx$ exists.

<u>Theorem 1</u>: Let $f_k : [a,b] \longrightarrow X_{\text{Banach}}$, f_k continuous, $\{f_k\} \to f$ <u>uniformly</u> on $[a,b]$, i.e. in $\|\cdot\|_{\sup}$.

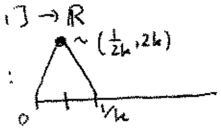
<u>Then</u>
$$\lim_{k\to\infty} \int_a^b f_k(s)\,ds = \int_a^b \lim_{k\to\infty} f_k(s)\,ds = \int_a^b f(s)\,ds$$

<u>proof</u>: $\int_a^b f(s)ds - \int_a^b f_k(s)ds =$

<u>pointwise convergence</u> does not suffice to interchange limits & integrals:

$f_k : [0,1] \to \mathbb{R}$

graph(f_k):



$f_k(t) \to 0 \ \forall t \in [0,1]$

$\int f_k(t)dt = 1$

<u>Example</u> : Let $L : X \to X$ Banach cont. & linear ($\|L\|_{op} < \infty$).

Consider the constant operator IVP on $\mathbb{R}$:

$$\text{IVP}\begin{cases} \frac{dx}{dt} = L(x) & t \in \mathbb{R} \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

(if $X = \mathbb{R}^n$, $L(x) = A_{n\times n}x$ and this is the homog. const coeff sys. you studied in 2280)

a) Following the CMP for <u>local</u> sol'ns of IVP's,
set $x_0(t) \equiv 0$.
Using $x_{k+1}(t) = x_0 + \int_0^t L(x_k(s))\,ds$

Show $\vec{x}_k(t) = (I + tL + \frac{t^2}{2!}L^2 + \dots + \frac{t^k}{k!}L^k)\,x_0$

b) In your hw. you showed that (for fixed t), $S_k(t) = \sum_{j=0}^{k} \frac{t^j}{j!} L^j$
is Cauchy in $(\mathcal{L}(X,X), \|\cdot\|_{op})$
so has a limit in the complete spce $\mathcal{L}(X,X)$, which we call e^{tL}

Thus
$\{x_k(t)\} \to e^{tL}x_0 := x(t)$
$\forall t$.

Local $\exists!$ says $e^{tL}x_0$
is the local unique sol'n to IVP.
Is it global sol'n?

(Notice, if $X = \mathbb{R}^n$, $L(x) = Ax$,
then $\|L\|_{op} = \max_{|x|=1} |Ax|$.
so (setting $x = e_j$), $\|L\|_{op} \geq \max|col_j(A)|$
$\geq |A|_\infty$
($= \max_{i,j} |a_{ij}|$).
so $\{S_k(t)\}$ Cauchy in $\|\cdot\|_{op}$
$\Rightarrow$ the matrices for $S_k(t)$,
$\sum_{j=0}^{k} \frac{t^j}{j!} A^j$ are converging
uniformly to the matrix e^{tA})

c) Use Theorem 1 to prove that $\forall t$, $\quad x(t) = x_0 + \int_0^t Lx(s)\,ds \quad$ I.E.

so that it also solves IVP (and so is unique sol'n).

- Start by showing that for $|s| \le M$,

$\{x_k(s)\} \to x(s)$ uniformly, so that you can apply theorem 1.

pf: $\quad x_k(s) - x(s) = \left(S_k(s) - e^{sL}\right) x_0$

$$\| S_k(s) - e^{sL} \|_{op} \le \sum_{j=k+1}^{\infty} \frac{|s|^j}{j!} \|L\|_{op}^j \le \sum_{j=k+1}^{\infty} \frac{M^j}{j!} \|L\|_{op}^j$$

$\to 0$ as $k \to \infty$
since Taylor series for e^z converges $\forall z$

- Now theorem 1:

$$x_{k+1}(t) = x_0 + \int_0^t \underbrace{L x_k(s)}\, ds$$

$\downarrow$ $Lx(s)$ uniformly for $|s| \le t$

$k \to \infty$

$$x(t) = x_0 + \int_0^t Lx(s)\,ds \qquad \blacksquare$$

Theorem 2 : Let $f_k : [a,b] \to X_{\text{Banach}}$, $\; f_k'(t)$ continuous.

Suppose $\{f_k\} \to f$ uniformly

and

$\{f_k'\} \to g$ uniformly.

Then f' exists and $f'(t) = \lim_{k\to\infty} f_k'(t) \qquad (= g'(t))$.

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proof. FTC $\Rightarrow \quad f_k(t) = f_k(a) + \int_a^t f_k'(s)\,ds$

let $k \to \infty$: $\downarrow \quad \downarrow \quad \downarrow$ by Theorem 1!

$$f(t) = f(a) + \int_a^t g(s)\,ds$$

then FTC $\Rightarrow f' \exists$, $f'(t) = g(t) \qquad \blacksquare$

Returning to example

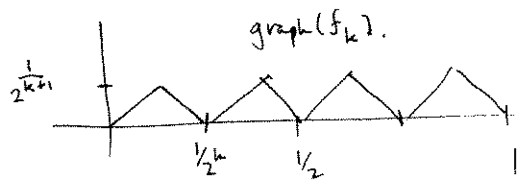
IVP $\begin{cases} x'(t) = L(x(t)) \\ x(0) = x_0 \end{cases}$

I.E. $x(t) = x(0) + \int_0^t L(x(s))\,ds$

Can you use Theorem 2 to show that the $x_k(t) = \left(\sum_{j=0}^{k} \frac{t^j}{j!} L^j\right) x_0$

converge to a sol'n $\vec{x}(t)$ of IVP?

example of why $\{f_k\} \to f$ uniformly does not imply $f_k' \to f'$:



$f_k \to 0$ uniformly on $[0,1]$.

$f_k' \to 0' = 0$. (In fact $f_k'(t) \not\to 0$ for any t!)