Math 5210
Wednesday Feb. 25

- Finish long time existence, uniqueness discussion in Tuesday notes

- There's actually a local existence (but not uniqueness) theorem for the IVP

$$\text{IVP} \begin{cases} \frac{d\vec{x}}{dt} = \vec{G}(t, \vec{x}(t)) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

which just requires at G be continuous (and bounded) on $Q = [t_0-\delta, t_0+\delta] \times \overline{B_R(x_0)}$
i.e. no Lipschitz est in the $\vec{x}$ variable is required.
The proof uses Euler approximations and equicontinuity (§11.2).... and proof works for $X = \mathbb{R}^n$, not ∞-dim'l.)
Friday?

<u>Example</u> $X = \mathbb{R}^n$, $A_{n\times n}$ constant matrix ... this example leads into §7.6, where we go next

$$\text{IVP} \begin{cases} \frac{d\vec{x}}{dt} = A\vec{x} \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

a) using initial fcn $\vec{x}_0(t) \equiv \vec{x}_0$
compute the Picard (CMP) iterates
$$\vec{x}_k(t) = x_0 + \int_0^t A\vec{x}_{k-1}(s)\, ds$$

ans: $\vec{x}_k(t) = (I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^k}{k!}A^k)\vec{x}_0$

b) Show the matrix valued sequence of functions $\{S_k(t)\} = \{\sum_{j=0}^{k} \frac{t^j}{j!} A^j\}$
is Cauchy in the operator norm,
and that the Cauchy estimates are uniform for $|t| \le C$.
(Since $\|B\|_\infty \le \|B\|_{op}$ the sequence is also Cauchy in $\|\cdot\|_\infty$).

Thus $\{S_k(t)\} \to$ a limit matrix, defined to be e^{tA}, uniformly for $|t| \le C$.

c) Thus $\vec{x}_k(t) = S_k(t)\,\vec{x}_0 \to e^{tA}\,\vec{x}_0$ uniformly for $|t| \le C$.

Call this limit $\vec{x}(t)$.

d) Does $\vec{x}(t)$ solve IVP $\forall t$? (i.e. not just for the small interval $[-\delta_1, \delta_1]$ guaranteed by contraction mapping principal)

Here's the setup:

$$
\begin{array}{ccc}
x_{k+1}(t) = & x_0 + \underbrace{\displaystyle\int_0^t A\,x_k(s)\,ds} \\
\Big\downarrow {\scriptstyle k\to\infty} & ?? \Big\downarrow {\scriptstyle k\to\infty} \\
x(t) & x_0 + \displaystyle\int_0^t A\,x(s)\,ds.
\end{array}
$$

(If ?? holds, then $x(t)$ solves I.E.
$$x(t) = x_0 + \int_0^t A\,x(s)\,ds$$
so solves IVP
$$\begin{cases} x' = Ax \\ x(0) = x_0 \end{cases}$$
)

?? follows from "interchange limit & integral" theorem:

<u>Theorem 1</u>: Let $f_k : [a,b] \to X_{\text{Banach}}$, f_k continuous, $\{f_k\} \to f$ <u>uniformly</u> on $[a,b]$, i.e. in $\|\cdot\|_{\sup}$.

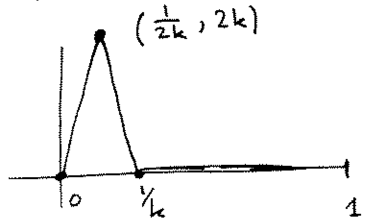
<u>Then</u>
$$\lim_{k\to\infty} \int_a^b f_k(s)\,ds = \int_a^b \lim_{k\to\infty} f_k(s)\,ds$$
$$= \int_a^b f(s)\,ds$$

<u>proof</u>: $\displaystyle\int_a^b f(s)\,ds - \int_a^b f_k(s)\,ds =$

example of why <u>pointwise convergence</u> not sufficient to interchange limit & integral:

$f_k : [0,1] \to \mathbb{R}$

graph (f_k)



for all fixed $t \in [0,1]$, $\{f_k(t)\} \to 0$

but

$$1 = \int_0^1 f_k(t)\,dt \not\to \int_0^1 \lim_{k\to\infty} f_k(t)\,dt = \int_0^1 0\,dt = 0.$$