Math 5210

Tuesday February 24 : 11.1–11.4

Do the local $\exists!$ theorem for 1^{st} order systems IVP, Monday notes. (This is akin to §11.4 in text). ~ there are typos on page 3, need $\delta_1 \leq R/M$ not M/R.

How do you piece together local solt'ns to get global ones?

$$\text{IVP} \begin{cases} \frac{d\vec{x}}{dt} = \vec{G}(t, \vec{x}(t)) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

$t_0 \in I \subset \mathbb{R}$.

<u>Theorem</u> : Consider the IVP above. Assume $\vec{G}(t,\vec{x})$ is defined $\forall\, t \in I$, $\vec{x} \in \vec{X}$. (and continuous)

A) If there is a global Lipschitz constant L so that
$$|\vec{G}(t,\vec{x}) - \vec{G}(t,\vec{z})| \leq L\,|x-z| \quad \forall x,z \in X$$
then $\exists!$ sol'n to IVP defined on <u>all</u> of I.

B) If the conditions of (A) don't hold globally, but do hold in a neighborhood of $(t_0, \vec{x}_0)$, then Define $t_1 := \sup\{t \geq t_0 \text{ s.t } \exists! \text{ sol'n to IVP on } [t_0, t]\}$.

Then either $\lim_{t \to t_1^-} \vec{x}(t)$ DNE ($\vec{x}(t)$ is the ! sol'n)

or if $\lim_{t \to t_1^-} \vec{x}(t) := \vec{x}(t_1)$ does exist, then local $\exists!$ conditions to not hold at $(t_1, \vec{x}(t_1))$.

(analogous result to the left of t_0)

<u>Examples</u>

A) The linear system in $\vec{X} = \mathbb{R}^n$ $\quad \frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{b}(t)$

With the $n \times n$ matrix $A(t)$ continuous, and continuous inhomogeneous term $\vec{b}(t)$, on any interval $I \subset \mathbb{R}$.

$$|\vec{G}(t,\vec{x}) - \vec{G}(t,\vec{z})| = |A(t)(\vec{x}-\vec{z})| \leq \|A\|_{op}\,|x-z|$$

has uniform Lipschitz bound on any compact subinterval, so $\exists!$ sol'n on any compact subinterval, so $\exists!$ sol'n on I.

B) Consider $\begin{cases} \frac{dx}{dt} = x^2 \\ x(0) = 1 \end{cases}$ on $I = \mathbb{R}$.

sol'n: $\frac{dx}{x^2} = dt$

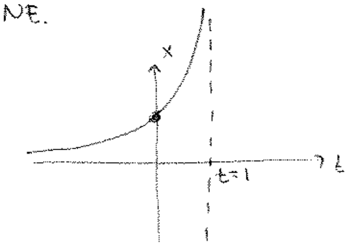
$-\frac{1}{x} = t + C$

@ $t=0$ $-1 = C$

$\frac{1}{x} = -t + 1$; $\boxed{x = \frac{1}{1-t}}$

(no uniform Lipschitz const for $G(x)$, since $G'(x) = 2x$ is unbounded).

$\lim_{t \to 1^-} x(t)$ DNE.



also consider

$$\begin{cases} \frac{dx}{dt} = 3x^{2/3} \\ x(0) = 1 \end{cases} \quad \text{on } I = \mathbb{R}$$

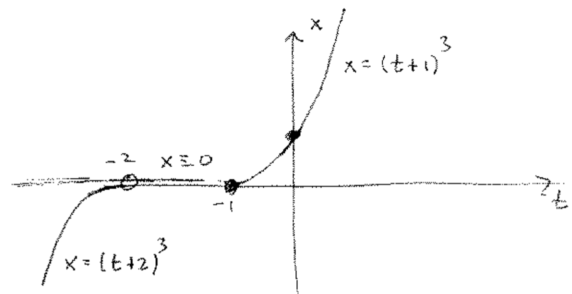
sol'n $\frac{1}{3} x^{-2/3} dx = dt$

$x^{1/3} = t + C$

$x = (t+C)^3$

$x(0) = 1 \Rightarrow x(t) = (t+1)^3$

but, don't forget the singular soltn $x(t) \equiv 0$.



one of the non-unique soltns to IVP

As t decreases from $t_0 = 0$, uniqueness fails first at $t = -1$, because $G(x) = 3x^{2/3}$ is not Lipschitz continuous in any nbhd of $x = 0$ ($G_x = 2x^{-1/3}$ is unbd. for x near 0).

<u>proof of page 1 theorem:</u>

...

Notice that if $\vec{x}(t)$ is the unique solution to

$$\begin{cases} \frac{d\vec{x}}{dt} = \vec{G}(t, \vec{x}(t)) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

on an interval $[t_0, t_1)$

and if $|\vec{G}(t, \vec{x}(t))| \leq M$ on $[t_0, t_1)$,

then $\lim_{t \to t_1^-} \vec{x}(t) := \vec{x}(t_1)$ exists

$t_1 - \delta < t_1 - \varepsilon$

$\left(|\vec{x}(t_1-\delta) - \vec{x}(t_1-\varepsilon)| \leq \left| \int_{t_1-\delta}^{t_1-\varepsilon} G(s, \vec{x}(s))\, ds \right| \leq M|\delta - \varepsilon| \right)$.

and if $\vec{G}(t, \vec{x})$ has local Lipschitz control in the $\vec{x}$ variable, in a neighborhood of $(t_1, \vec{x}(t_1))$, then for $\varepsilon > 0$ sufficiently small, the local solution to

$$\begin{cases} \frac{d\vec{z}}{dt} = \vec{G}(t, \vec{z}(t)) \\ \vec{z}(t_1-\varepsilon) = x(t_1-\varepsilon) \end{cases}$$

will agree with $\vec{x}(t)$, and extend it beyond $t = t_1$

proof of A: Assume not true.

Let $t_1 = \sup\{ t \geq t_0 \text{ s.t. } \exists! \text{ sol'n to IVP on } [t_0, t) \}$.

By the discussion on the previous page, and the estimate

$$|G(t, x(t))| \leq G(t, 0) + L\,|x(t)|$$
$$\leq C + L\,|x(t)| \quad \forall\, t \in [t_0, t_1].$$

We deduce $\lim_{t \to t^-} |x(t)| = \infty$.

But this is impossible: for $t < t_1$

$$x(t) = x_0 + \int_{t_0}^{t} G(s, x(s))\,ds$$

$$x(t + \Delta t) = x(t) + \int_{t}^{t+\Delta t} G(s, x(s))\,ds$$

$\Delta t > 0$: $$\frac{|x(t+\Delta t)| - |x(t)|}{\Delta t} \leq \frac{1}{\Delta t} \int_t^{t+\Delta t} |G(s, x(s))|\,ds \leq \frac{1}{\Delta t} \int_t^{t+\Delta t} M + L|x(s)|\,ds$$

$M = \max\{ |G(s,0)|,\ s \in [t_0, t_1] \}$.

deduce $|x(t)|' \leq M + L\,|x(t)|$

(assume $|\cdot|$ is diff'ble, $|\cdot| : X \to \mathbb{R}$, except maybe at $x = 0$, then this estimate is being made on subintervals with $x \neq 0$)

$w(t) := |x(t)|$

$$w' - Lw \leq M$$
$$(e^{-Lt} w)' \leq e^{-Lt} M$$
$$e^{-Lt} w \Big]_{t_0}^{t} \leq \int_{t_0}^{t} M e^{-Ls}\,ds = \frac{M}{L}\left(e^{-Lt_0} - e^{-Lt}\right) \leq \frac{M}{L} e^{-Lt_0}$$
$$e^{-Lt} w(t) - e^{-Lt_0} w(t_0) \leq \frac{M}{L} e^{-Lt_0}$$
$$w(t) \leq e^{L(t-t_0)} w(t_0) + \frac{M}{L} e^{L(t-t_0)} \quad \text{is bounded on } [t_0, t_1]$$

∎ A

proof of B: If $\lim_{t \to t_1^-} \vec{x}(t) := \vec{x}(t_1)$ exists, and if local $\exists!$ hypotheses did hold, you would extend $\vec{x}(t)$ onto a longer interval (uniquely).