Math 5210

Monday Feb 23 : First do chain rule & Lipschitz est. from old notes

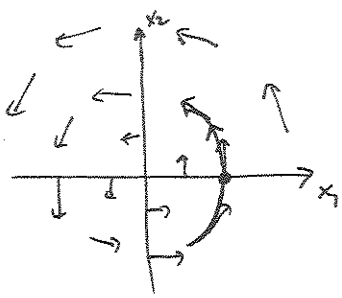
Existence and Uniqueness for 1^{st} order systems of DE's.

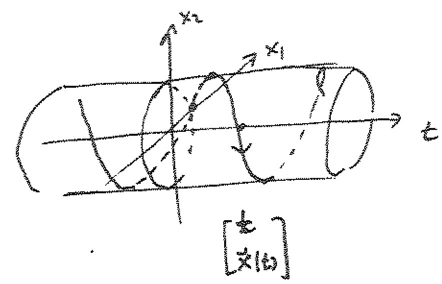
$$\text{IVP} \begin{cases} \frac{d\vec{x}}{dt} = \vec{G}(t, \vec{x}(t)) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

<u>example</u>

$$\begin{cases} y'' + y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases} \qquad \begin{matrix} x_1(t) := y(t) \\ x_2(t) := y'(t) \end{matrix} \quad \Rightarrow \quad \begin{cases} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{cases}$$

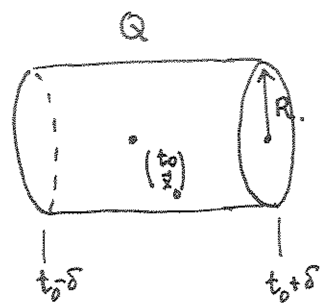
$\Leftarrow \quad y(t) := x_1(t)$





<u>General set up : local $\exists !$ thm.</u>

$$\text{IVP} \begin{cases} \frac{d\vec{x}}{dt} = \vec{G}(t, \vec{x}(t)) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$



- G continuous on
  $Q(x_0, R, t_0, \delta) = [t_0 - \delta, t_0 + \delta] \times \overline{B_R(x_0)}$ $\rightarrow$ recall HW on product metric spaces.
  $\overline{B_R(x_0)} \subset X$ Banach.
- $\|G\|_{sup} \le M$ on Q.

  also

  - $\|G(t,x) - G(t,y)\| \le L\|x-y\| \quad \forall (t,x), (t,y) \in Q.$ i.e. G is (uniformly) Lipschitz in the X variable, with t fixed.

Then if $\delta_1 < \min\left(\delta, \frac{1}{L}, \frac{R}{M}\right)$

$\exists !$ soln to IVP on $[t_0 - \delta_1, t_0 + \delta_1]$

Lemma : $\vec{x}(t)$ is a differentiable soltn to IVP on an interval I containing t_0

iff

$\vec{x}(t)$ is a continuous soltn to IE on the same interval.

IVP $\begin{cases} \frac{d\vec{x}}{dt} = G(t, x(t)) \\ x(t_0) = x_0 \end{cases}$ IE : $x(t) = x_0 + \int_{t_0}^{t} G(s, x(s))\, ds.$

pf :

proof of $\exists!$ thm :

Let $\widetilde{\mathcal{F}} = \left\{ z : [t_0-\delta_1, t_0+\delta_1] \to \overline{B_R(x_0)} \text{ s.t. } z(t_0) = x_0,\ z \text{ cont},\ \|\cdot\|_{sup} \right\}.$

- $\widetilde{\mathcal{F}}$ is a complete metric space :

$\{z_k\}$ Cauchy in $\widetilde{\mathcal{F}} \Rightarrow \{z_k(t)\}$ Cauchy in X, has limit $:= z(t)$.
(t fixed)

if $k, \ell \geq M \Rightarrow \|z_k - z_\ell\|_{sup} < \varepsilon$ then $\|z_k(t) - z_\ell(t)\| < \varepsilon$

$\downarrow \ell \to \infty$

$\|z_k(t) - z(t)\| \leq \varepsilon \quad \forall t,\ k \geq m$

$\Rightarrow \|z_k - z\|_{sup} \to 0$

old HW $\Rightarrow z \in \widetilde{\mathcal{F}}$, i.e. cont.

Here's an contraction map: (we hope)

$$T : \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$$

$$Tz(t) := x_0 + \int_{t_0}^{t} G(s, z(s))\, ds$$

(Notice, if $x \in \tilde{\mathcal{F}}$ is a fixed point . then it solves IE, so it solves IVP)

- $T : \tilde{\mathcal{F}} \to \mathcal{F}$.

  Need $\| Tz(t) - x_0 \| \le R \quad \forall\, t \in [t_0 - \delta_1, t_0 + \delta_1]$.

  if it exists for t in original interval, define t_1 s.t. $\| Tz(t) - x_0 \| = R$ and $|t_0 - t_1|$ is minimal.

  Then
  $$Tz(t_1) - x_0 = \int_{t_0}^{t_1} G(s, z(s))\, ds$$

  $$\Rightarrow \underset{\overset{\shortparallel}{R}}{\| 7_0 \|} \le |t_0 - t_1| M \qquad (\text{~~what~~ why?})$$

  $$\Rightarrow |t_0 - t_1| \geq \frac{M}{R}. \qquad * \text{ take } \delta_1 \le M/R.$$

- T a contraction:

  $$Tx(t) - Ty(t) = \int_{t_0}^{t} G(s, x(s)) - G(s, y(s))\, ds$$

  $$\Rightarrow \|\cdot\| \le \left| \int_{t_0}^{t} \|\cdot\|\, ds \right| \le \left| \int_{t_0}^{t} L \underbrace{\|x(s) - y(s)\|}_{\le \|x-y\|_{sup}} ds \right. \le \underbrace{L\,|t - t_0|}_{\substack{\text{if } \le \mu < 1 \\ \text{get contraction.}}} \|x - y\|_{sup}$$

  $** \text{ take } \delta_1 < \frac{1}{L}$

  $$\Rightarrow \| Tx - Ty \|_{sup} \le \underbrace{\left(\frac{\delta_1}{L}\right)}_{<1.} \|x - y\|_{sup}$$

So, for $\delta_1 < \min\left(M/R, \frac{1}{L}, \delta\right)$

$T : \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$ is a contraction

$\Rightarrow \exists!$ fixed pt $\Rightarrow \exists!$ soln to IVP on this interval.

and, soln is $\|\cdot\|_{sup}$ limit of. $\{x_k(t)\}$

$$x_0(t) \equiv x_0$$
$$x_1 := T x_0$$
$$x_2 := T^2 x_0$$
$$\vdots$$
$$x_k := T^{(k)} x_0$$

$\boxtimes$

More fun HW!

Hints on #2: $L: \mathbb{R}^n \to \mathbb{R}^n$, $L(\vec{x}) = A_{n\times n}\vec{x}$

$$\|L\|_{op} = \sup_{|x|=1} \langle A\vec{x}, A\vec{x}\rangle \qquad (\langle\ ,\ \rangle \text{ for dot product})$$

$$= \max_{|x|=1} \langle A\vec{x}, A\vec{x}\rangle$$

$$\text{but } A\vec{x}\cdot A\vec{x} = \underset{\substack{\uparrow \\ \text{row vect}}}{(Ax)^T}\ \underset{\substack{\uparrow \\ \text{col vect}}}{Ax} = x^T(A^TA)x$$

A^TA is symmetric so has an orthonormal eigenbasis, by spectral theorem. you can check (as above), all eigenvalues of A^TA must be ≥ 0. Then express $\vec{x}$ as a linear combo of the A^TA-eigenvectors, and compute $A\vec{x}\cdot A\vec{x}$.

new problems:

Exercise 4 Now let X, Y Banach, define $\mathcal{L}(X,Y) = \{L: X\to Y \text{ linear, s.t. } \|L\|_{op}<\infty\}$.

note $\mathcal{L}(X,Y)$ is a vector space, if we define $c_1L_1+c_2L_2$ by

$$(c_1L_1+c_2L_2)(v) = c_1L_1(v)+c_2L_2(v)$$

4a) Prove $(\mathcal{L}(X,Y), \|\cdot\|_{op})$ is a normed vector space

4b) let $\{L_k\}$ Cauchy in $\mathcal{L}(X,Y)$.

prove that $L(v) := \lim_{k\to\infty} L_k(v)$ exists $\forall v\in V$, and is a linear operator with $\|L\|_{op}<\infty$.

Then show $\{L_k\}\to L$ in $(\mathcal{L}(X,Y), \|\cdot\|_{op})$, so this space is complete, i.e. Banach.

Exercise 5 Examples of 4b): let $L\in\mathcal{L}(X,X)$.

5a) If $\|L\|_{op}<1$, and $I: X\to X$ is the identity map,

prove $\sum_{k=0}^{\infty} L^k$ converges, i.e. the partial sums are Cauchy. $(L^0 = I)$

prove that this sum actually equals $(I-L)^{-1}$ (and satisfies $\|I-L\|_{op} \leq \frac{1}{1-\|L\|_{op}}$

5b) If $L\in\mathcal{L}(X,X)$ then

$$e^L := \sum_{k=0}^{\infty} \frac{L^k}{k!} \text{ exists, and } \|e^L\|_{op} \leq e^{\|L\|_{op}}.$$