Math 5210
Friday Feb 20

HW for Friday Feb. 27
Class Exercises 1,2,3
11.1.6 # 2c, 3, 5

We'll work through Wednesday's notes... I've added comments/details in today's.

- Details of uniqueness of the differential map, <u>page 1 Wed</u>:

Let $f(x+h) - f(x) = L_1(h) + R_1(h)$ as in the defn of diffble.
$\qquad\qquad\qquad\quad = L_2(h) + R_2(h)$

Pick any $h \in X$, $|h| = 1$, $0 < t \leq 1$:

$$L_1(th) + R_1(th) = L_2(th) + R_2(th)$$

$$L_1(th) - L_2(th) = R_2(th) - R_1(th)$$

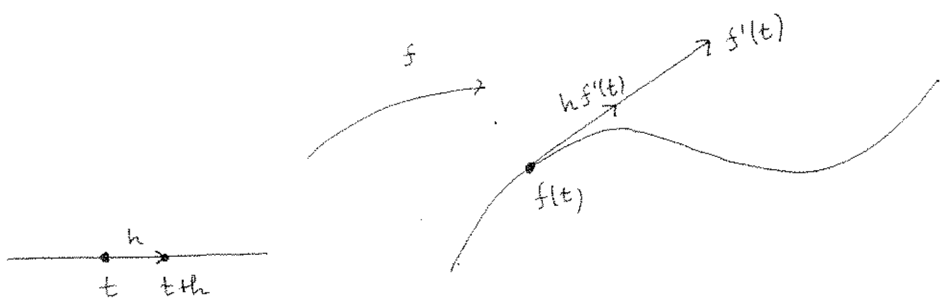
$\div t \qquad L_1(h) - L_2(h) = \frac{1}{t} R_2(th) - \frac{1}{t} R_1(th)$

$\lim_{t \to 0}: \qquad L_1(h) - L_2(h) = 0 \quad$, since $\frac{|R_i(th)|}{|th|} \to 0$ as $t \to 0$.

Thus $L_1(h) = L_2(h) \quad \forall\, h$ s.t. $|h| = 1$.
But for $v \neq 0$, $\quad L_1(v) = |v|\, L_1\!\left(\frac{v}{|v|}\right) = |v|\, L_2\!\left(\frac{v}{|v|}\right) = L_2(v)$

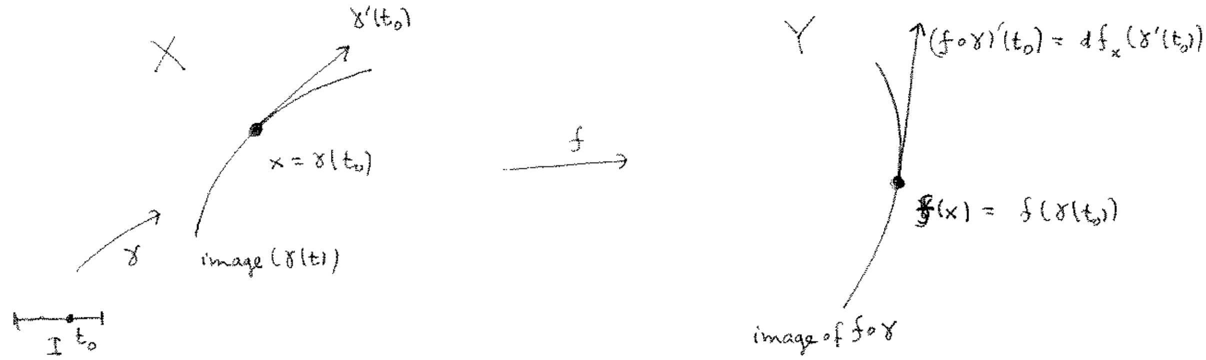
so $L_1 = L_2$ $\blacksquare$

- I was sloppy on page 2 1
The differential map is $h \longmapsto f'(t)\, h$
i.e. the <u>matrix</u> $[df_t]$ of the differential map is the tangent vector $f'(t)$.



continuing page 2 comments:

3. A good way to understand the differential map: $f : X \to Y$



$$df_x : T_x X \longrightarrow T_{f(x)} Y$$

Then $df_x(\gamma'(t_0)) = (f \circ \gamma)'(t_0)$.

<u>proof</u>: by the approximation formula,

$$f(\gamma(t_0+\Delta t)) = f(\gamma(t_0)) + df_x(\overbrace{\gamma(t_0+\Delta t) - \gamma(t_0)}^{:=h}) + R(h)$$

$$\frac{1}{\Delta t}\Big(f(\gamma(t_0+\Delta t)) - f(\gamma(t_0))\Big) = df_x\left[\frac{\gamma(t_0+\Delta t) - \gamma(t_0)}{\Delta t}\right] + \frac{R(h)}{\Delta t}$$

used linearity of df_x

take $\lim_{\Delta t \to 0}$!; $\frac{R(h)}{\Delta t} \to 0$.

∎

Class Exercises

Exercise 1 Let X, Y normed vector spaces

$L : X \to Y$ linear. $\left(L(sx_1 + tx_2) = sL(x_1) + tL(x_2) \quad \forall x_1, x_2 \in X,\ s,t \in \mathbb{R} \right)$

Define $\|L\|_{op} := \sup_{|x| \le 1} |L(x)|$

1a) Prove $\|L\|_{op} = \sup_{|x|=1} |L(x)|$

1b) Prove that $\forall x \in X$, $|L(x)| \le \|L\|_{op} |x|$

1c) If $M : Y \to Z$ is also linear,

prove $\|M \circ L\|_{op} \le \|M\|_{op} \|L\|_{op}$

Exercise 2 a) If $L : \mathbb{R}^n \to \mathbb{R}^n$, $L(\vec{x}) = A\vec{x}$

use the spectral theorem (for symmetric matrices), (2270)

to prove $\|L\|_{op} = \left(\text{max eigenvalue of } A^T A\right)^{1/2}$

2b) Find $\|A\|_{op}$ for $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

Exercise 3 Let $L : X \to Y$ linear. (X, Y normed vector spaces)

a) Prove L is continuous ($\forall x \in X$) iff L is continuous at $x = 0$.

b) Prove L is continuous at $x=0$ iff $\|L\|_{op} < \infty$.