Math 5210
Monday Febrarary 2

- Finish the discussion of interior, closure, boundary, as in page 3 Wednesday notes
- Subspaces of metric spaces, and relatively open and closed sets, from Friday's notes.

Then,
<u>compactness</u>! (to be continued Tuesday...)

<u>Definition</u> $A \subset X$ is <u>compact</u> iff $\forall \{x_k\} \subset A$ $\exists$ subseq $\{x_{k_j}\} \to x \in A$.

<u>Example</u>: $A \subset \mathbb{R}$ is compact iff A is closed and bounded. (Same result holds in $\mathbb{R}^n$)

pf $\Rightarrow$: show A is closed, i.e. contains its limit points:
let $\{a_k\} \subset A$, $\{a_k\} \to x \in \mathbb{R}$.
compactness $\Rightarrow \exists \{a_{k_j}\} \to a \in A$.
subseq's of conv. seqs converge to same limits, so $a = x \in A$
so A closed ∎

show A is bounded.
If not, $\exists a_k \in A$, $|a_k| > k$
$$\Rightarrow \lim_{l \to \infty} |a_k - a_l| \geq \left(\lim_{l \to \infty} |a_l|\right) - a_k = +\infty$$
so no subsequence is Cauchy, so no subseq is convergent. ∎

$\Leftarrow$: If A is closed and bounded then $\exists N < \infty$, $A \subset [-N, N] = I_0$
let $\{x_k\} \subset A$ a sequence. Choose $I_1 = [-N, 0]$ or $[0, N]$ s.t. $\#(\{x_k\} \cap I_1) = \infty$.
Choose $x_{k_1} \in I_1$.
Choose I_2 left or right half of I_1 s.t. $\#(\{x_k\} \cap I_2) = \infty$
$x_{k_2} \in I_2$, $k_2 > 1$
inductively construct $\{x_{k_j}\}$, $m, l \geq j \Rightarrow x_{k_m}, x_{k_l} \in I_l \Rightarrow |x_{k_l} - x_{k_m}| < \frac{N}{2^j} \Rightarrow$ Cauchy
$\mathbb{R}$ complete $\Rightarrow \{x_{k_j}\} \to x \in \mathbb{R}$
A closed $\Rightarrow x \in A$ ∎

Could you see how to do this in $\mathbb{R}^n$? For example, in $\mathbb{R}^2$?

in general (X,d) compactness of subsets is not quite equivalent to closed and bounded:

Thm Let $A \subset X$, (X,d) metric.
If A is compact, then

(1) A is <u>complete</u> (every Cauchy seq in A converges, to an elt of A)
in particular, A contains its limit points and is <u>closed</u>

(2) A is <u>totally bounded</u> : $\forall \varepsilon > 0$ $\exists$ finitely many points $\{z_1, \dots z_{N_\varepsilon}\}$ s.t.

$$A \subset \bigcup_{j=1}^{N_\varepsilon} \underbrace{B(z_j, \varepsilon)}_{B_\varepsilon(z_j)}$$

↑ stronger than bounded in general; in $\mathbb{R}^n$ it's equivalent.

Conversely, if (1) & (2) hold then A is compact.

pf: (1) Let $\{x_k\} \subset A$ Cauchy. A compact $\Rightarrow \exists$ subseq. $\{x_{k_j}\} \to a \in A$
$\Rightarrow \{x_k\} \to a$ ∎

(2) Let $\varepsilon > 0$. Pick $z_1 \in A$
if $A \not\subset B_\varepsilon(z_1)$ pick $z_2 \in A \cap (B_\varepsilon(z_1))^c$
if $\bigcup_{j=1}^{2} B_\varepsilon(z_j) \not\supset A$ pick z_3 in $A \setminus \bigcup_{j=1}^{2} B_\varepsilon(z_j)$

induct. If process does not end after a finite # of steps
you've constructed $\{z_k\} \subset A$, $d(z_k, z_\ell) > \varepsilon$ $\forall k, \ell$.
such sequences have no convergent subsequences $\Rightarrow\!\Leftarrow$

Hence, at some N we have $A \subset \bigcup_{j=1}^{N} B_\varepsilon(z_j)$ ∎

<u>Converse</u> Let A be complete and totally bounded.
We use the divide and conquer idea from page 1 to prove compact:

Let $\{x_k\} \subset A$. For each $n \in \mathbb{N}$ we can cover A with finitely many $\frac{1}{n}$-balls:

$$A \subset \bigcup_{\ell=1}^{N_n} B(z_\ell^n, \tfrac{1}{n})$$

Important note:
if A is totally bd the collection of centers $\{z_\ell^n\}_{n \in \mathbb{N},\ \ell = 1,2,\dots N_n}$ is a countable dense subset of A!

for $n=1$ pick $B_1 = B(z_\ell^1, 1)$ s.t. $\#(\{x_k\} \cap B_1) = \infty$
~~and~~ i.e. $\exists \{x_{k_j}\} \subset B_1$

for $n=2$ pick $B_2 = B(z_\ell^2, \frac{1}{2})$ s.t. $\#(\{x_{k_j}\} \cap B_2) = \infty$
i.e. $\exists \{x_{k_{j_s}}\} \subset B_2$.

induct. Now make a subsequence of the original $\{x_k\}$ by picking the 1^{st} elts of each subsequence above, i.e. $\{x_{k_1}, x_{k_{j_1}}, x_{k_{j_{s_1}}}, \dots\} = \{y_\ell\}$. then $k, \ell \geq m \Rightarrow d(y_k, y_\ell) < \frac{1}{m}$
Cauchy.

$\Rightarrow \{y_\ell\} \to a \in A$ ∎