Math 5210
Wednesday Feb. 18

- Finish differentiable curves and FTC, for curves into Banach Spaces, pages 4-6 Tuesday notes
- Then discuss differentiability for maps between Banach Spaces:

(because for the ODE system $\frac{d\vec{x}}{dt} = \vec{f}(t,\vec{x})$, $f: \underset{\mathbb{R}}{\overset{}{I}} \times X \to X$ is such a map) ($I \subset \mathbb{R}$)

let X, Y Banach.

$x \in \mathcal{O}_{open} \subset X$, $f: \mathcal{O} \to Y$

Def f is <u>differentiable</u> at x iff $\exists$ a <u>linear</u> map $L: X \to Y$, with $\|L\|_{op} < \infty$

(recall 2270)

such that for $x+h \in \mathcal{O}$,

$$f(x+h) = f(x) + L(h) + R(h)$$

$$\text{where } \lim_{h\to 0} \frac{|R(h)|}{|h|} = 0$$

(recall from HW,
$\|L\|_{op} := \sup\{|L(x)|, \text{ s.t. } |x| \le 1\}$
$(= \sup\{|L(x)|, \text{ s.t. } |x| = 1\})$
In HW next week you'll show that L is continuous iff $\|L\|_{op} < \infty$)

- L is unique:

suppose $f(x+h) - f(x) = L_1(h) + R_1(h)$
$\qquad\qquad\qquad\quad = L_2(h) + R_2(h)$

then

Since L is unique (and for other reasons), we write df_x for it.
(df_x is called the <u>differential</u> or <u>derivative map</u>, at x, since it transforms tangent vectors h based at $x \in X$ to tangent vectors $df_x(h)$ based at $f(x) \in Y$)

$$\boxed{f(x+h) = f(x) + df_x(h) + R_x(h)}$$

(will sometimes write $R(x,h)$)

Examples (3220)

① $f: \mathbb{R} \supset I \to X$, $t \in I$. Then we just defined f differentiable at t (Tuesday)

$$\text{iff} \quad f'(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} \quad \text{exists}$$

This is equivalent to

$$f'(t) - \left(\frac{f(t+h)-f(t)}{h}\right) := \varepsilon_t(h) \to 0 \quad \text{as } h \to 0$$

i.e. $$f(t+h) = f(t) + \underset{\in X}{f'(t)}\,\underset{\text{scalar}}{h} + \underbrace{\varepsilon_t(h)h}_{:= R_t(h)}$$

where $\left|\frac{R_t(h)}{h}\right| = |\varepsilon_t(h)| \to 0$ as $h \to 0$

$$\boxed{df_t = f'(t)}$$

② $f: \mathbb{R}^n \supset \mathcal{O} \to \mathbb{R}^m$, $x \in \mathcal{O}$. If f is differentiable at x, then for displacements $\vec{h} = \underset{\text{scalar}}{h}\vec{e}_j$,

$$f(x+h\vec{e}_j) = f(x) + df_x(h\vec{e}_j) + R_x(h\vec{e}_j)$$

$$\frac{1}{h}\left[f(x+he_j) - f(x)\right] = df_x(\vec{e}_j) + \frac{1}{h}R_x(h\vec{e}_j)$$

$$\Rightarrow \lim_{h\to 0} \frac{1}{h}\left[\vec{f}(\vec{x}+h\vec{e}_j) - \vec{f}(\vec{x})\right] = df_x(\vec{e}_j) + 0 \quad \text{exists.}$$

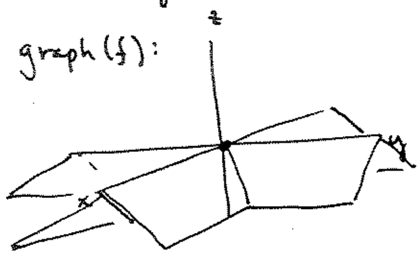
$\Rightarrow \dfrac{\partial \vec{f}}{\partial x_j}$ exists $\left(= \begin{bmatrix} \frac{\partial f_1}{\partial x_j} \\ \vdots \\ \frac{\partial f_m}{\partial x_j} \end{bmatrix} \right)$ and the matrix for df_x with respect to the standard bases in $\mathbb{R}^n$ & $\mathbb{R}^m$ is just the Jacobian matrix

$$\left[\frac{\partial f_i}{\partial x_j}\right]_{\substack{i=1\ldots m \\ j=1\ldots n}}.$$

The converse to ② is not true (partials $\exists \not\Rightarrow$ diff'ble) e.g. consider the roof fcn $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = -\min(|x|,|y|).$$

graph(f):



$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0.$

So *if* f diff'ble there,

$$f\binom{h_1}{h_2} = 0 + 0 + R_0\left(\binom{h_1}{h_2}\right), \quad \frac{|R|}{|h|} \to 0 \text{ as } h \to 0$$

in particular,

$$\frac{\left|f\binom{h}{h}\right|}{|h|} \to 0 \text{ as } h \to 0$$

$$\underset{1!}{\shortparallel} \qquad \Rightarrow\!\Leftarrow.$$

But:

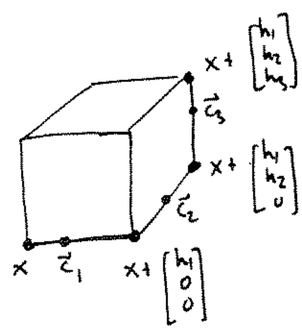
<u>Theorem</u> $f: \mathcal{O} \to \mathbb{R}^m$, $\mathbb{R}^n \supset \mathcal{O} \ni x$. If all partial derivatives of f exist and are continuous in a nbhd of x <u>then</u> f is diff'ble at x. (In our text see e.g. 10.1)

<u>proof.</u> for $h = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$ define $h^{(k)} = \begin{bmatrix} h_1 \\ \vdots \\ h_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. ($h^{(0)} := \vec{0}$).

Then

$$f(x+h) - f(x) = \sum_{k=1}^{n} \underbrace{f(x+h^{(k)}) - f(x+h^{(k-1)})}_{= h_k \frac{\partial f}{\partial x_k}(\vec{c}_k)} \quad \text{(telescopes).}$$

by 1-var. Mean Value thm!



So

$$f(x+h) - f(x) = \underbrace{\sum_{k=1}^{n} h_k \frac{\partial f}{\partial x_k}(\vec{x})}_{\left[\frac{\partial f_i}{\partial x_j}\right]\vec{h}} + \underbrace{\sum_{k=1}^{n} \left[\frac{\partial f}{\partial x_k}(c_k) - \frac{\partial f}{\partial x_k}(\vec{x})\right] h_k}_{R(x,h)}\ ;$$

$$|R(x,h)| \le \left\| \left[\frac{\partial f_i}{\partial x_k}(c_k) - \frac{\partial f_i}{\partial x_k}(0)\right] \right\|_{op} |h|$$

$\downarrow$

0 as $|h| \to 0$.

sorry! notation switch for $R_x(h)$

$(\|A\|_{op} \le \|A\|_2)$

Chain Rule X, Y, Z Banach spaces.

$f: X \to Y$

$g: Y \to Z$

if f is diff'ble at x

& g is diff'ble at $f(x)$

then $g \circ f$ is diff'ble at x and

$$\boxed{d(g \circ f)_x = [dg_{f(x)}] \circ [df_x].}$$

pf $f(x+h) = f(x) + \underbrace{df_x(h) + R_f(x,h)}_{:=k}$

$$g(f(x)+k) = g(f(x)) + dg_{f(x)}(k) + R_g(k).$$

$$\Rightarrow g(f(x+h)) = g(f(x)) + dg_{f(x)} \circ \left(df_x(h) + R_f(x,h)\right) + R_g(k)$$

$$= g(f(x)) + (dg_{f(x)} \circ df_x)(h) \quad + \underbrace{dg_{f(x)} \circ R_f(x,h)}_{\mathrm{I}} + \underbrace{R_g(k)}_{+\mathrm{II}}$$

this has operator norm $\leq \|dg_{f(x)}\|_{op} \|df_x\|_{op}$ → hw for next week

$$\frac{|\mathrm{I} + \mathrm{II}|}{|h|} \leq \frac{|\mathrm{I}|}{|h|} + \frac{|\mathrm{II}|}{|h|} ;$$

$$\frac{|\mathrm{I}|}{|h|} \leq \|dg_{f(x)}\|_{op} \frac{|R_f(x,h)|}{|h|}$$

$\downarrow 0$ as $h \to 0$.

$\frac{|R_f(x,h)|}{|h|} < 1.$

$\downarrow$

$\varepsilon = 1$

if $k=0$, $R_g(k) = 0$.

Else, $\dfrac{|R_g(k)|}{|h|} = \dfrac{|R_g(k)|}{|k|} \dfrac{|k|}{|h|}$.

$|k| \leq (\|df_x\|_{op} + 1)|h|$ for $|h| < \delta$

so $h \to 0 \Rightarrow k \to 0$

$\dfrac{|R_g(k)|}{|k|} \to 0$ as $h \to 0$

$\dfrac{|k|}{|h|}$ bd by $\|df_x\|_{op} + 1$

$$\Rightarrow \frac{|\mathrm{I}+\mathrm{II}|}{|h|} \to 0 \text{ as } h \to 0 \quad \blacksquare$$

<u>Example</u>. Let $\underset{\Theta}{\overset{X}{\cup}}$ open and <u>convex</u>

$$\downarrow$$

$$x, z \in \Theta \Rightarrow \{(1-t)x + tz,\ 0 \le t \le 1\} \subset \Theta.$$

Let $f : \Theta \to Y$ diff'ble $\forall x \in \Theta$,

$$\|df_x\|_{op} \le M \quad \forall x.$$

then $|f(x) - f(z)| \le M|x-z| \quad \forall x, z \in \Theta$

[in this case we say that f is <u>Lipschitz</u> <u>continuous</u> with Lipschitz constant M. This is an important kind of uniform continuity].

<u>proof</u> $\quad f(x) - f(z) = \displaystyle\int_0^1 \frac{d}{dt} f(tx + (1-t)z)\, dt \quad$ by FTC and chain rule.

(<u>note</u> if $\varphi : \underset{I}{\overset{\mathbb{R}}{\cup}} \to X$ diff'ble, f as above, then

$$\frac{d}{dt}(f \circ \varphi) = d(f \circ \varphi)_t(1) \quad \text{by example 1 page 2}$$

$$= df_{\varphi(t)} \cdot d\varphi_t(1)$$

$$= df_{\varphi(t)} \circ \varphi'(t)$$

).