Math 5210
Tuesday Feb. 17

HW for Friday Feb 20

Exercise 1a, 1b : Show $\mathcal{K} = \{K, d_H\}$ is a complete metric space when the underlying space (X, d) is complete. (Feb 11 notes)

Exercise 2 today's notes: Lemma to prove FTC (b), for Banach space valued fcns.

that's it for this week!

We are heading to §11.1, to prove the $\exists!$ theorem for 1st order systems of DE's (as an application of the contraction mapp

$$\begin{cases} \frac{d\vec{x}}{dt} = \vec{f}(t, \vec{x}(t)) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

Except we shall prove this theorem in the generality of Banach space target.
(Dynamical system PDE's are often considered as ODE's where
$u_t = u_{xx}$
$u_{tt} = c^2 u_{xx}$ + more

$t \longmapsto u(\cdot, t)$ is a function of x (living in a Banach space) for each t.)

Thus we need to redo some Calculus in this setting!

The Riemann Integral

Theorem : let $(X, \|\cdot\|)$ a normed vector space, complete (i.e. Banach space)

$f : [a,b] \longrightarrow X$ continuous.
$[a,b] \subset (\mathbb{R}, |\cdot|)$

Let P be a partition choice: $a = x_0 < x_1 < \cdots < x_n = b$

$x_i^* \in I_i = [x_{i-1}, x_i]$, $\Delta x_i = x_i - x_{i-1}$.

$$\mathcal{R}_P := \sum_{i=1}^{n} f(x_i^*) \Delta x_i \qquad \|P\| := \max(\Delta x_i)$$

Then $\boxed{\lim_{\|P\| \to 0} \mathcal{R}_P := \int_a^b f(x)\,dx \in X \quad \text{exists.}}$ (no fussing with upper & lower sums).

Furthermore, $\boxed{\left\| \int_a^b f(x)\,dx \right\| \leq \int_a^b \|f(x)\|\,dx .}$ ← Text calls this "Minkowski ineq"

Also, if $X = \mathbb{R}^n$, and $\vec{f}(x) = \begin{bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{bmatrix}$, then $\int_a^b \vec{f}(x)\,dx = \begin{bmatrix} \int_a^b f_1(x)\,dx \\ \vdots \\ \int_a^b f_n(x)\,dx \end{bmatrix}$, as expected.

②

<u>Lemma</u>: $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $\|P\|, \|Q\| < \delta \Rightarrow \|R_P - R_Q\| < \varepsilon$

<u>proof</u> $f : [a,b] \to X$ continuous $\Rightarrow f$ uniformly continuous (why?)

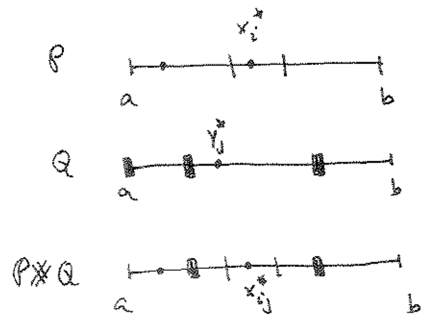
let $\varepsilon > 0$. pick δ s.t. $|x-y| < \delta \Rightarrow \|f(x) - f(y)\| < \dfrac{\varepsilon}{2(b-a)}$

let $P = \bigcup_{i=1}^{n} I_i \ni x_i^*$, $\quad Q = \bigcup_{j=1}^{m} J_j \ni y_j^*$

$\|P\|, \|Q\| < \delta$.

Define the refinement

$$P \circledast Q = \bigcup_{\substack{i,j \\ I_i \cap J_j \neq \emptyset}} (I_i \cap J_j) \ni x_{ij}^*$$



$$R_P - R_{P \circledast Q} = \sum_i f(x_i^*) \Delta x_i - \sum_{\substack{i,j \\ I_i \cap J_j \neq \emptyset}} f(x_{ij}^*) \Delta x_{ij}$$

$$= \sum_i \left[f(x_i^*) \Delta x_i - \sum_{\substack{j \text{ s.t.} \\ I_i \cap J_j \neq \emptyset}} f(x_{ij}^*) \Delta x_{ij} \right] \qquad \text{note, } \sum_{\substack{j \text{ s.t.} \\ I_i \cap J_j \neq \emptyset}} \Delta x_{ij} = \Delta x_i$$

$$= \sum_i \sum_{\substack{j \text{ s.t.} \\ I_i \cap J_j \neq \emptyset}} \left(f(x_i^*) - f(x_{ij}^*) \right) \Delta x_{ij}$$

$$\Delta \text{ ineq} \Rightarrow \|R_P - R_{P \circledast Q}\| \leq \sum_i \sum_{\substack{j \text{ s.t.} \\ I_i \cap J_j \neq \emptyset}} \overbrace{\| \quad \|}^{< \varepsilon / 2(b-a)} \Delta x_{ij} < \frac{\varepsilon}{2(b-a)} (b-a) = \varepsilon/2$$

similarly, $\|R_Q - R_{P \circledast Q}\| < \varepsilon/2$

$$\Rightarrow \|R_P - R_Q\| < \|R_P - R_{P \circledast Q} + (R_{P \circledast Q} - R_Q)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

∎

proof of Theorem

Pick partition P_k, $\|P_k\| < 1/k$.

Lemma $\Rightarrow \{R_{P_k}\}$ Cauchy in X, so it has a limit $:= \int_a^b f(x)\,dx$ in this complete space.

Now let $\varepsilon > 0$. Pick δ s.t. $\begin{matrix}\|P\| \\ \|Q\|\end{matrix} < \delta \Rightarrow \|R_P - R_Q\| < \varepsilon/2$

and $\frac{1}{k} < \delta$, and s.t. $\| R_{P_k} - \int_a^b f(x)\,dx | < \varepsilon/2$

then $\|P\| < \delta \Rightarrow \| R_P - \int_a^b f(x)\,dx | \le \| R_P - R_{P_k} \| + \| R_{P_k} - \int_a^b f(x)\,dx \|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

$$\boxed{\text{so } \lim_{\|P\| \to 0} R_P := \int_a^b f(x)\,dx \text{ exists}}$$

missing details:
prove the estimate: $\| \int_a^b f(x)\,dx \| \le \int_a^b \|f(x)\|\,dx$ very important!!! (application of Δ ineq.)

Finally, if $X = \mathbb{R}^n$

$$f(\vec{x}) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

$$\Rightarrow R_P(f) = \begin{bmatrix} R_P(f_1) \\ R_P(f_2) \\ \vdots \\ R_P(f_n) \end{bmatrix} \xrightarrow{\text{as } \|P\| \to 0} \begin{bmatrix} \int_a^b f_1(x)\,dx \\ \vdots \\ \int_a^b f_n(x)\,dx \end{bmatrix}$$

, since in $\mathbb{R}^n$ convergence of all components is equivalent to convergence of vectors.

$\blacksquare$

Usual consequences of Riemann sum def of integrals ...

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

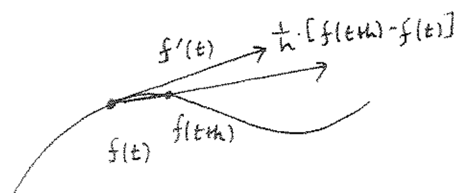
$$\left[\int_b^a f(x)\,dx := -\int_a^b f(x)dx \text{ for } a<b, \quad \int_a^a f(x)dx := 0 \right]$$

$$\int_a^b c_1 f(x) + c_2 g(x)\,dx = c_1 \int_a^b f(x)\,dx + c_2 \int_a^b g(x)\,dx$$

<u>differentiable curves and F.T.C</u>

$I \subset \mathbb{R}$ an interval.

$f : I \to (X, \|\cdot\|)$ (Banach) <u>differentiable</u> iff $\forall\, t \in I$, $\lim_{\substack{h\to 0 \\ t+h \in I}} \frac{1}{h}\left(f(t+h) - f(t)\right) =: f'(t)$ exists.



<u>example</u>: in $\mathbb{R}^n$, $f(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$

$$\frac{1}{h}\left[f(t+h) - f(t)\right] = \frac{1}{h}\begin{bmatrix} f_1(t+h) - f_1(t) \\ f_2(t+h) - f_2(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{f_1(t+h) - f_1(t)}{h} \\ \vdots \\ \frac{f_n(t+h) - f_n(t)}{h} \end{bmatrix}$$

so $f'(t)\ \exists$ iff $\begin{bmatrix} f_1' \\ f_2' \\ \vdots \\ f_n' \end{bmatrix} \exists$, and in this case they are equal.

Fundamental Theorem of Calculus

(a) let $g: I \to X$ continuous, $a \in I$

then $G(t) := \int_a^t g(s)\,ds$ is differentiable on I

and $G'(t) = g(t)$

(b) let $f: I \to X$ differentiable, $f'(t)$ continuous. let $[a,b] \subset I$

Then $\int_a^b f'(t)\,dt = f(b) - f(a)$.

proof if $X = \mathbb{R}^n$ is easy, assuming you know FTC for $X = \mathbb{R}$:

But it's a little Harder if X is (an infinite dim'l) Banach space!

(a) $\frac{G(t+h) - G(t)}{h} = \frac{1}{h} \int_t^{t+h} g(s)\,ds = \frac{1}{h} \int_t^{t+h} g(t) + [g(s) - g(t)]\,ds$

now estimate, and prove (a):

h

(b) If $X = \mathbb{R}$ you usually use mean-value thm to do (b):

Let $F(t) = f(t) - f(a)$ Then $F'(t) = f'(t)$ Also $F(a) = 0 = G(a)$.

$G(t) = \int_a^t f'(s)\,ds$ $G'(t) = f'(t)$ by (a).

so $(F-G)'(t) \equiv 0$ and $(F-G)(a) = 0$

so $(F-G)(t) \equiv 0$ (MVT) $\rightarrow$ If $X = \mathbb{R}$

so $F(t) \equiv G(t)$

$F(b) = f(b) - f(a) = G(b) = \int_a^b f'(s)\,ds.$ $\blacksquare$

$\uparrow$ if $X = \mathbb{R}$.

No MVT in general X!
Everything works in X = Banach
if you can prove

<u>Lemma</u>: Let $Z : I \to X$ differentiable, $t_0 \in I$

$Z(t_0) = 0$

$Z'(t) = 0 \quad \forall t \in I$

Then $Z(t) \equiv 0 \quad \forall t \in I$

This is homework! <u>Exercise 2</u>

Hint: Let $[a,b] \subset I$ and let $\varepsilon > 0$
$\quad \ni t_0$

then $\forall t \in [a,b] \ \exists\ \delta > 0$ s.t. $|s-t| < \delta \Rightarrow \| \frac{1}{s-t}(Z(s) - Z(t)) \| < \varepsilon$

$\Rightarrow \|Z(s)\| \leq \|Z(t)\| + \varepsilon |t-s|$

Use this and a covering of the compact interval $[a,b]$ to prove

$$|Z(t)| \leq \varepsilon |t - t_0| \quad \forall t \in [a,b].$$

Then let $\varepsilon \to 0$, and exhaust I with compact subintervals!