Math 5210
Friday February 13

I'll post HW later today or Saturday.

future plans:

- 11.1 $\exists!$ sys. DE's
  - "review" integral $\int_a^b f(t)\,dt$, $f:[a,b]\to B$ Banach ... relates to 6.1–6.2
- 7.3 limits of integrals
- 7.5 Convolutions, Weierstrasse
- 7.6 Equicontinuity, Arzela-Ascoli
- 9.3 Finish

⋮

- prove Theorem 2 for iterated function system fractals, the fixed "points" in the Hausdorff metric space $\mathcal{K}$ $\left(\mathcal{K}=\{K\subset X,\ K \text{ compact}\}\text{, where the underlying space } (X,d) \text{ is complete}\right)$

  of the set mapping
  $$\tilde{\mathcal{F}}(K):=f_1(K)\cup f_2(K)\cup\cdots\cup f_\ell(K)$$
  for $f_j: X\to X$ a contraction, with contraction const. μ_j, $j=1\ldots n$

<u>proof</u>: Let $\mu=\max(\mu_1,\ldots,\mu_\ell)<1$.

We show
$$\tilde{\mathcal{F}}:\mathcal{K}\to\mathcal{K} \text{ is a contraction, with c.c. } \mu.$$

step 0): Why is $\tilde{\mathcal{F}}:\mathcal{K}\to\mathcal{K}$, i.e. if $K\subset X$ is compact, why is $\tilde{\mathcal{F}}(K)=\bigcup_{j=1}^{\ell} f_j(K)$ compact?

<u>rest of proof</u>:

Let $K_1, K_2\in\mathcal{K}$, $d_H(K_1,K_2)=\varepsilon$.

Thus $K_1\subset K_{2,\varepsilon}$
$K_2\subset K_{1,\varepsilon}$.

For $1\le j\le \ell$, we claim
$$f_j(K_1)\subset \left(f_j(K_2)\right)_{,\mu_j\varepsilon}$$
$$f_j(K_2)\subset \left(f_j(K_1)\right)_{,\mu_j\varepsilon}$$

check: Let $k_1\in K_1$. $\exists$ $k_2\in K_2$ s.t. $d(k_1,k_2)\le\varepsilon$
$\Rightarrow d(f_j(k_1), f_j(k_2))\le \mu_j\varepsilon$ ■

Thus,
$$\tilde{\mathcal{F}}(K_1)=\bigcup_{j=1}^{\ell} f_j(K_1)\subset\bigcup_{j=1}^{\ell}\left(f_j(K_2)\right)_{,\mu_j\varepsilon}$$
$$=\left(\tilde{\mathcal{F}}(K_2)\right)_{,\mu_j\varepsilon}$$

(and interchange K_1 & K_2 for symmetric result ■

<u>Remark</u>: We'll return to fractals like these when we discuss measure theory.

- It turns out that the <u>Hausdorff dimension</u> of fractals is usually not an integer.
  (i.e. it's a real number "fraction", usually not rational however).
  (e.g. the dimension of Sierpinski's triangle is $\frac{\ln 3}{\ln 2} \approx 1.58$)

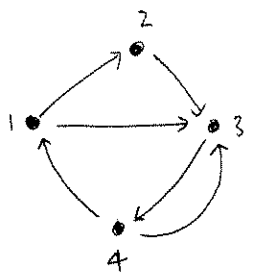
Another contraction mapping principle application: google page ranking.

http://www.ams.org/featurecolumn/archive/pagerank.html

<u>Example 1</u>: a giving (voting) game.

Consider 4 people who like to give to their friends
sites who link other sites

according to this directed graph.



- Assume you give equally to each friend you link to.

Start with initial fractions of "vote," $\vec{x}_0 = \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \\ x_{0,4} \end{bmatrix}$ $\quad x_{0,j} \geq 0$, $\sum_{j=1}^{n} x_{0,j} = 1.$

Keep track of successive amounts

$$\vec{x}_{k+1} = x_{k,1} \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix} + x_{k,2} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_{k,3} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_{k,4} \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}$$

vote vector at stage $k+1$.

#1's current vote (fraction)

#2's current vote

$$\vec{x}_{k+1} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{B} \vec{x}_k \quad ; \quad \vec{x}_0 = \text{initial vote.}$$

discrete dynamical system with constant transition matrix

so, $\vec{x}_k = B^k \vec{x}_0$

Does $\lim_{k \to \infty} B^k \vec{x}_0$ exist?

Does it depend on $\vec{x}_0$?

If the limiting vector exists, then it says who ends up with the most stuff
~ highest page rank.

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> with(linalg):
  Digits:=6:
> B:=matrix(4,4,[0,0,0,.5,.5,0,0,0,.5,1,0,.5,0,0,1,0]);
```

$$B := \begin{bmatrix} 0 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

```
> evalm(B^50);
```

$$\begin{bmatrix} 0.181817 & 0.181817 & 0.181818 & 0.181818 \\ 0.0909089 & 0.0909096 & 0.0909084 & 0.0909093 \\ 0.363636 & 0.363635 & 0.363636 & 0.363635 \\ 0.363634 & 0.363636 & 0.363633 & 0.363637 \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \\ x_{0,4} \end{bmatrix}$$

<u>Def</u> $A_{n\times n}$ is a <u>stochastic</u> matrix iff

- $a_{ij} > 0 \quad \forall i,j$
- $\sum_{i=1}^{n} a_{ij} = 1 \quad \forall i = 1 \ldots n$ (column entries add up to 1)

<u>Def</u> $B_{n\times n}$ is almost stochastic if there is a power k so that $B^k = A$ is stochastic.

<u>Theorem</u> Let A be stochastic. $A_{n\times n}$

Let $X = \left\{ \vec{x} \in \mathbb{R}^n \text{ s.t. } \begin{array}{l} x_i \geq 0 \quad i = 1 \ldots n \\ \text{and } \sum_{i=1}^{n} x_i = 1 \end{array} \right\}$.

Define $T : X \to X$ by $T\vec{x} = A\vec{x}$.

Then, for the $p=1$ norm on X $\left(\|\vec{x}\| = \sum_{i=1}^{n} |x_i| \right)$,

T is a contraction.

Thus $\exists!$ fixed point $\vec{x}_g$ s.t. $T\vec{x}_g = \vec{x}_g$

for google.

(although this theorem is really the Perron-Frobenius thm).

and $\forall \vec{x}_0 \in X$, $T^n \vec{x}_0 \to \vec{x}_g$.

in particular,

$$\underbrace{T^n \vec{e}_j}_{col_j(A^n)} \to \vec{x}_g \quad , \text{ so } A^n \to \left[\vec{x}_g \middle| \vec{x}_g \middle| \cdots \vec{x}_g \right].$$

$$\vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \leftarrow \text{entry } j$$

<u>Cor</u> If B is almost stochastic, then $\exists! \, \vec{x}_g$ s.t. $B^n \vec{x}_0 \to \vec{x}_g \quad \forall x_0 \in X$.

pf:
$$\begin{aligned} B^n \vec{x}_0 &= B^{k\ell} B^j \vec{x}_0 \\ &= A^\ell (B^j \vec{x}_0) \\ &\to \vec{x}_g \text{ for } A, \text{ for all possible } B^j\vec{x}_0. \end{aligned}$$

<u>example</u> <u>page 2</u>

proof of Theorem

- first, note that $T: X \to X$,

  i.e. if the initial vote adds up to 1, so does the new vote (clear that all $x_j \geq 0 \Rightarrow$ same for all $(A\vec{x})_j$)

  this is just "conservation of vote",

  but algebraically,

  if $\sum_{i=1}^{n} x_i = 1$

  $$\text{then } \cancel{\sup} \sum_{i=1}^{n} (A\vec{x})_i = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_j = \sum_{j=1}^{n} x_j \underbrace{\sum_{i=1}^{n} a_{ij}}_{1} = \sum_{j=1}^{n} x_j = 1.$$

- T is a contraction on $(X, \|\cdot\|_1)$:

  $a_{ij} > 0 \ \forall i,j$, so $\exists\ \varepsilon > 0$ s.t. $a_{ij} \geq \varepsilon \ \forall i,j$.

  write $A = \varepsilon \mathbb{1} + C$, $[\mathbb{1}]_{ij} = 1 \ \forall i,j$, $c_{ij} \geq 0 \ \forall i,j$

  therefore $\sum_{i=1}^{n} a_{ij} = 1$

  $\Rightarrow \sum_{i=1}^{n} c_{ij} = 1 - \varepsilon n := \mu$. $\forall j = 1 \ldots n$.

Let $\vec{x}, \vec{y} \in X$:

$$\|A\vec{x} - Ay\|_1 = \|\cancel{\varepsilon \mathbb{1}(x-y)} + C(\vec{x}-\vec{y})\|_1$$

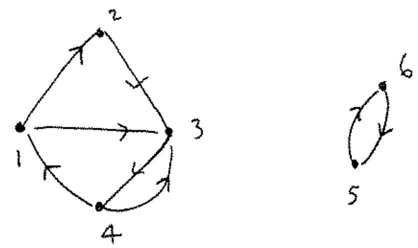
0

$$\mathbb{1}\vec{x} = \begin{bmatrix} \Sigma x_i \\ \Sigma x_i \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \mathbb{1}\vec{y}$$

$$\begin{aligned}
&= \|C(\vec{x}-\vec{y})\|_1 \\
&= \sum_{j=1}^{n} \left| \sum_{i=1}^{n} c_{ij}(x_i - y_i) \right| \\
&\leq \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}|x_i - y_i| = \sum_{i=1}^{n}\sum_{j=1}^{n} |x_i - y_i| c_{ij} \\
&= \sum_{i=1}^{n} |x_i - y_i| \underbrace{\sum_{j=1}^{n} c_{ij}}_{\mu} = \mu \|x - y\|_1
\end{aligned}$$

$\blacksquare$

Example 2



$$B = \left[\begin{array}{cccc|cc} 0 & 0 & 0 & .5 & & \\ .5 & 0 & 0 & 0 & & 0 \\ .5 & 1 & 0 & .5 & & \\ 0 & 0 & 1 & 0 & & \\ \hline & & 0 & & 0 & 1 \\ & & & & 1 & 0 \end{array}\right]$$

no B^k is stochastic!

Google fudge factor

$$A := \frac{\varepsilon}{n} \mathbb{1} + (1-\varepsilon) B$$ is stochastic (google $\varepsilon = .2$ or $.3$)

for $\varepsilon = .3$, $\frac{\varepsilon}{6} = .05$:

$$A := \begin{bmatrix} 0.05 & 0.05 & 0.05 & 0.400 & 0.05 & 0.05 \\ 0.400 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0.400 & 0.75 & 0.05 & 0.400 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.75 & 0.05 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.75 \\ 0.05 & 0.05 & 0.05 & 0.05 & 0.75 & 0.05 \end{bmatrix}$$

```
> evalm(A^20);
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$$\begin{bmatrix} 0.124920 & 0.124919 & 0.124922 & 0.124919 & 0.124775 & 0.124775 \\ 0.0937297 & 0.0937313 & 0.0937293 & 0.0937308 & 0.0936574 & 0.0936574 \\ 0.234269 & 0.234268 & 0.234271 & 0.234268 & 0.233978 & 0.233978 \\ 0.214016 & 0.214020 & 0.214016 & 0.214019 & 0.213726 & 0.213727 \\ 0.166534 & 0.166535 & 0.166535 & 0.166535 & 0.167332 & 0.166534 \\ 0.166534 & 0.166535 & 0.166534 & 0.166534 & 0.166534 & 0.167332 \end{bmatrix}$$

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