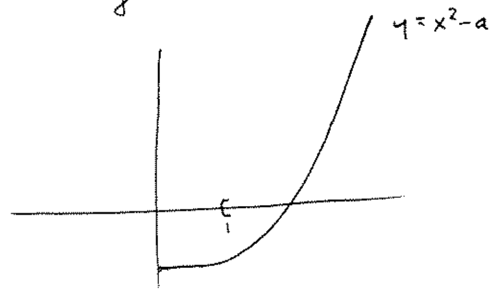
Math 5210
Wednesday Feb. 11

- Contraction mapping principle ~ yesterday's notes

- 9.3 #16 is an example of Newton's method leading to contraction mappings.
 (although the x-domain should be the closed interval $[1, \infty)$)
 Here we are using Newton's method to find a sol'n to $g(x)=0$, $g(x)=x^2-a$
 i.e. $x=\sqrt{a}$.
 Check:



- If $g: \mathbb{R}^n \to \mathbb{R}^n$ and you search for a root $g(z)=0$,
 there is also a newton's method, using the affine approximation for g:

$$g(x_{n+1}) \approx g(x_n) + \underset{\substack{\uparrow \\ \text{the derivative matrix}}}{g'(x_n)}(x_{n+1}-x_n)$$

set $$0 = g(x_n) + g'(x_n)(x_{n+1}-x_n) \to -g(x_n) = g'(x_n)(x_{n+1}-x_n)$$
$$[-g'(x_n)]^{-1} g(x_n) = x_{n+1}-x_n$$
$$x_{n+1} = x_n - [g'(x_n)]^{-1} g(x_n)$$

i.e. the iteration map is $\boxed{T(x) = x - [g'(x)]^{-1} g(x)}$ and you can show that if g is continuously differentiable and you start at an x_0 with $|g(x_0)|$ close enough to zero, then T is a contraction is a nbhd, so that Newton iteration converges to a root z of $g(z)=0$

This circle of ideas is related to constructive proofs of the inverse fcn theorem.

Hausdorff distance on compact sets

(leads to idea of contraction maps which construct fractals)

Let (X, d) be complete.

Let $\mathcal{K} = \{ K \subset X$ s.t. K is compact$\}$.

Distance on $\mathcal{K}$!

First, define <u>closed ε-nbhds of K</u>:

$$K_{,\varepsilon} := \{ x \in X \text{ s.t. } \exists\, k \in K \text{ with } d(x,k) \leq \varepsilon \}$$

(If you need glasses, K probably looks like some $K_{,\varepsilon}$)

If K_1, K_2 are compact, and for some small $\varepsilon > 0$,

$$K_2 \subset K_{1,\varepsilon} \quad \text{and} \quad K_1 \subset K_{2,\varepsilon}$$

then K_1, K_2 are "close"... leads to

$$d_H(K_1, K_2) := \inf \{ \varepsilon \text{ s.t. } K_2 \subset K_{1,\varepsilon} \text{ and } K_1 \subset K_{2,\varepsilon} \}$$

$$= \min \{ \varepsilon \text{ s.t. } K_2 \subset K_{1,\varepsilon} \text{ and } K_1 \subset K_{2,\varepsilon} \}$$

Hausdorff distance

pf: call ε_0 the infimum.

Let $k_2 \in K_2$.

Then for $\varepsilon = \varepsilon_0 + \frac{1}{j}$ $\exists\, h_j \in K_1$

i.e. $d(k_2, h_j) \leq \varepsilon_0 + \frac{1}{j}$, since $K_2 \subset K_{1, \varepsilon_0 + \frac{1}{j}}$

K_1 compact, so

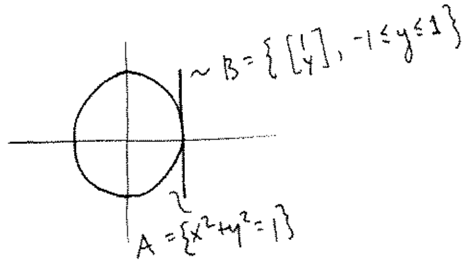
$\exists$ subseq. $\{h_{j_\ell}\} \to h \in K_1$

$$d(k_2, h_{j_\ell}) \leq \varepsilon_0 + \tfrac{1}{j}$$
$$\downarrow$$
$$d(k_2, h) \leq \varepsilon_0 \implies k \in K_{1,\varepsilon_0}$$

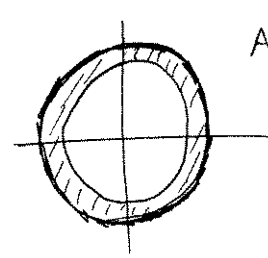
Thus $k \in K_{2,\varepsilon}$ $\forall \varepsilon > \varepsilon_0$ $\implies k \in K_{2,\varepsilon_0}$ ∎

<u>Examples</u> What are the Hausdorff distances between these pairs of sets?

$\mathbb{R}^2$



$\mathbb{R}^2$



$A = \{ x^2+y^2 = 1 \}$

$B = \{ .9 \leq \sqrt{x^2+y^2} \leq 1 \}$.

<u>Theorem 1</u> If (X,d) is complete, $\mathcal{K} = \{K \subset X \text{ s.t. } K \text{ compact}\}$

Then

a) $(\mathcal{K}, d_H)$ is a metric space

b) $(\mathcal{K}, d_H)$ is complete.

i.e. if $\{K_j\}$ is Cauchy $\exists\ K \in \mathcal{K}$ s.t.

$\{K_j\} \to K$.

(in fact, $K = \{x \in X \text{ s.t. } \forall \varepsilon > 0\ \exists m \text{ s.t. } j \geq m \Rightarrow x \in K_{j,\varepsilon}\}$).

<u>pf</u> HW 9.3.29, for next week.

a) is pretty straightforward

b) may challenge you.

<u>Theorem 2</u>: Let (X,d) complete, $(\mathcal{K}, d_H)$ as above

Let $\{f_1, f_2 .. f_\ell\}$ be a finite collection of contraction mappings,

$f_j : X \to X$, with contraction consts, $0 \leq \mu_j < 1$.

Let $\mu := \max\{\mu_1, \mu_2, .. \mu_\ell\}$.

Then the mapping

$\tilde{\mathcal{F}} : \mathcal{K} \to \mathcal{K}$

$\tilde{\mathcal{F}}(K) := f_1(K) \cup f_2(K) \cup .. \cup f_\ell(K)$

is a contraction on $(\mathcal{K}, d_H)$, with contraction constant μ.

Thus $\tilde{\mathcal{F}}$ has a unique fixed point, and it is the limit

of $\{\tilde{\mathcal{F}}^j(K)\}$, for any initial set $K \in \mathcal{K}$.

(we'll check Thm 2 on Friday.)

for examples
~ see notes!