Math 5210
Tuesday Feb. 10.

Connectivity + Contraction Mappings

this was the theorem we did not complete in class yesterday. →

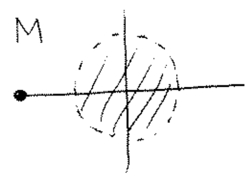
> extra hw problem for Friday:
>
> Expand 9.3 #10, to show that $(C(X,Y), d)$ as defined in Monday's notes, is complete metric space. Here X is compact and Y is complete. Hint: follow the template from the special case $(C([a,b]), \|\cdot\|_\infty)$ from jan27-pdf. (#10 is one of the steps)

Def: Let M be a metric space.
M is connected iff $\nexists$ disjoint non-empty open sets A, B, with $M = A \cup B$ ← such an A, B are called a disconnection

i.e. iff the only sets in M which are both open and closed are M and $\emptyset$.

Example the subspace of $\mathbb{R}^2$ given below is not connected: $M = \left\{ \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\} \cup B_1\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$
Explain.

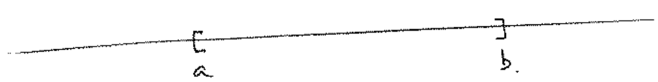
M



Theorem 1 Let $M \subset \mathbb{R}$. Then M is connected iff M is an interval

pf. $\Leftarrow$: Let M an interval (bd or unbounded, containing some of its endpoints or not.)

Let $A \neq \emptyset$, A open and closed. (in M).
We show $B := A^c$ is $\emptyset$.
If not, pick $a \in A$, $b \in A^c$. Assume $a < b$ w.o.l.o.g.
Since M is an interval, $[a,b] \subset M$



divide and conquer:

$a_0 = a \qquad b_0 = b$

if $\frac{a_0+b_0}{2} \in A$, let $a_1 = \frac{a_0+b_0}{2}$, $b_1 = b_0$

if $\frac{a_0+b_0}{2} \in A^c$ let $a_1 = a_0$, $b_1 = \frac{a_1+a_0}{2}$

if $a_j < b_j$, $a_j \in A$, $b_j \in A^c$

if $\frac{a_j+b_j}{2} \in A$ then $a_{j+1} = \frac{a_j+b_j}{2}$, $b_{j+1} = b_j$.

" " $\notin$ " $b_{j+1} =$ " , $a_{j+1} = a_j$.

$\{a_k\} \nearrow$
$\{b_k\} \searrow$ $\quad |b_k - a_k| = \frac{|b-a|}{2^k} \to 0$ as $k \to \infty$

$\ell \geq k \Rightarrow |a_\ell - a_k|, |b_\ell - b_k| \leq \frac{|b-a|}{2^k}$

so $\{a_k\} \nearrow a \swarrow \{b_k\}$. $\Rightarrow a \notin \mathring{A} = A$
and $a \notin \mathring{A^c} = A^c$ $\quad \Rightarrow\!\Leftarrow$ $\qquad$ Thus intervals are connected.

$\Rightarrow$: Let $M \subset \mathbb{R}$ connected. We show M is an interval.

It suffices to show that whenever $\alpha < \beta$, $\alpha \in M$ and $\beta \in M$, then $[\alpha, \beta] \subset M$.

(This property characterizes intervals.
For example, if M is bounded, then let $a = \inf M$, $b = \sup M$.
Then if M has the interval property, and since $\exists \{a_k\} \subset M$, $\{a_k\} \searrow a$, $\{b_k\} \subset M$, $\{b_k\} \nearrow b$,

deduce $\underbrace{\bigcup_k [a_k, b_k]}_{= (a, b)} \subset M$. Thus $M = (a,b), [a,b), (a,b]$ or $[a,b]$.

unbounded cases analogous.)

So, suppose $\exists$ $\alpha < \beta$, $\alpha \in M$, $\beta \in M$ but $\alpha < \gamma < \beta$, $\gamma \notin M$.

then $M \cap (-\infty, \gamma) = A$ and $M \cap (\gamma, \infty) = B$ is a disconnection $\times$

thus the subinterval property holds and M is an interval.

<u>Theorem 2</u> Let X connected, $f: X \to Y$ continuous.
Then $f(X)$ is connected.

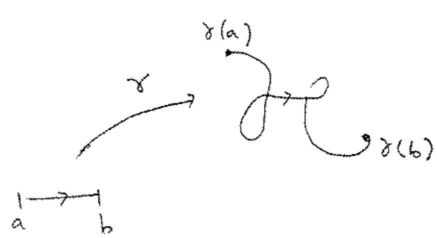
pf:

<u>Corollary</u>: (Intermediate Value Theorem): Let $f: X \to \mathbb{R}$, X connected
If $\exists$ x_1, x_2 with $f(x_1) < f(x_2)$
then $\forall$ $f(x_1) < c < f(x_2)$ $\exists$ $x \in X$ s.t. $f(x) = c$.

pf:

<u>Silly example</u> prove that at any instant $\exists$ antipodal points on earth where the temperatures are the same.

<u>Def</u> A <u>path</u> in X is a continuous map $\gamma : \underbrace{[a,b]}_{\subset \mathbb{R}} \longrightarrow X$



<u>Def</u> X is path connected iff $\forall x, z \in X$
$\exists$ a path $\gamma : [a,b] \longrightarrow X$ with
$\gamma(a) = x$
$\gamma(b) = z$.

<u>Theorem 3</u> X path connected $\Rightarrow$ X connected.

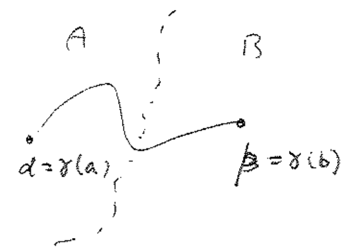
(Reverse implication true for open subsets of normed vector spaces, but not in general, see HW).

pf : Assume X is not connected.
Let $X = A \cup B$ with A, B, open, disjoint, non-empty.

pick $\alpha \in A$, $\beta \in B$

$\gamma : [a,b] \to X$ a path

$\gamma(a) = \alpha$
$\gamma(b) = \beta$.

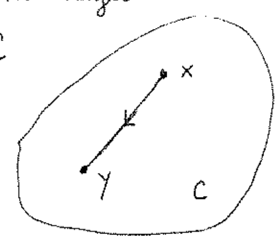


Then $\gamma([a,b]) \cap A$, $\gamma([a,b]) \cap B$ is a disconnection of $\gamma([a,b])$. This violates Theorem 2. $\times$ ∎

<u>Example</u>

<u>Def</u> A subset C of a vector space is convex iff $\forall\, x, y \in C$ the path images

$\gamma(t) = (1-t)x + ty \in C$
$\forall\ 0 \le t \le 1$



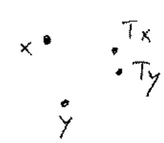
By Theorem 3, convex subsets of normed vector spaces are connected.

This is one way to deduce $\mathbb{R}^n$ is connected (h.w. from last week), although a direct divide & conquer attack would've worked too.

<u>New topic</u>: <u>Contraction mapping theorem</u> (We'll see many examples).

<u>Def</u> A mapping $T : X \to X$ is a <u>contraction</u> iff $\exists\ 0 \le \mu < 1$ s.t. $d(T(x), T(y)) \le \mu\, d(x,y)$ $\forall\, x, y \in X$.

(write



<u>Theorem</u> Let $T : X \to X$ a contraction mapping, X complete. Then T has a unique <u>fixed point</u> $x \in X$, i.e. s.t. $T(x) = x$.

<u>proof</u>: Let X complete, and T a contraction mapping with shrink factor $0 \le \mu < 1$.
T can have at most one fixed point, since if $T(x) = x$ and $T(z) = z$

then $d(T(x), T(z)) \le \mu\, d(x,z)$
$d(x,z) \le \mu\, d(x,z)$
$(1-\mu)\, d(x,z) \le 0$
so $d(x,z) = 0$ ∎

Here's how to find a fixed point:

Pick (any) $x_0 \in X$

$$x_1 := T(x_0)$$
$$x_2 := T(x_1) = T \circ T(x_0) = T^2(x_0)$$
$$x_{k+1} := T(x_k) = T^k(x_0)$$

$$d(x_2, x_1) = d(T(x_1), T(x_0)) \leq \mu\, d(x_1, x_0)$$

$$d(x_3, x_2) = d(T(x_2), T(x_1)) \leq \mu\, d(x_2, x_1) \leq \mu^2 d(x_1, x_0)$$

by induction,

$$d(x_{k+1}, x_k) \leq \mu^k d(x_1, x_0)$$

so for $\ell > k$,

$$d(x_\ell, x_k) \leq \sum_{j=k}^{\ell-1} d(x_j, x_{j+1}) \leq \sum_{j=k}^{\ell-1} d(x_1, x_0)\mu^j \leq d(x_1, x_0) \sum_{j=k}^{\infty} \mu^j$$
$$= d(x_1, x_0) \frac{\mu^k}{1-\mu} \to 0 \text{ as } \ell > k \to \infty.$$

Thus $\{x_k\}$ Cauchy.

X complete $\Rightarrow \{x_k\} \to x \in X$

T cont $\Rightarrow \{T(x_k)\} \to T(x)$

$$\{T(x_k)\} = \{x_{k+1}\} \to x$$

Thus $T(x) = x$ ∎

This theorem seems fairly abstract, but has the most amazing concrete applications.