Math 5210
Wednesday April 8

- pages 3-4 Tuesday, for Hausdorff measure & dimension

- Theorem 1 : Let μ be an outer measure on X.
  Let $\tilde{\mathcal{F}} = \{E \subset X$ s.t. E is measurable$\}$.
  Then $\tilde{\mathcal{F}}$ is a σ-algebra.
  Furthermore, for any $A \subset X$, and $\{E_k\} \subset \tilde{\mathcal{F}}$ disjoint,
  $$\mu\left(A \cap \left(\bigcup_k E_k\right)\right) = \sum_k \mu(A \cap E_k).$$
  proof: We did this (thinking we were dealing with $\mu = m^*$, and Lebesgue measurable sets) in two stages. First (p.5 March 30) show the measurable sets form a Boolean algebra, then (p.1-2 Apr 1), show $\tilde{\mathcal{F}}$ is a σ-algebra, and the countable additivity.

  consider it an uncollected class exercise to go back through this theorem.

- Theorem 2 : Let μ be a metric outer measure. Let $\tilde{\mathcal{F}}$ be the μ-measurable subsets of X.
  Then the Borel σ-algebra $\mathcal{B} \subset \tilde{\mathcal{F}}$.

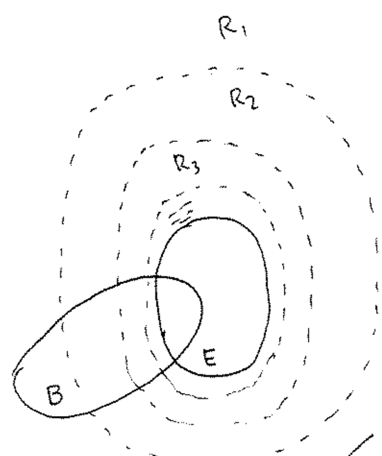
  pf: below (and in Strichartz).

Corollary : If μ is a metric outer measure, then μ restricted to $\mathcal{B}$, or μ restricted to $\tilde{\mathcal{F}}$, is a measure.

proof of Theorem 2

It suffices to prove that if $E \subset X$ is closed, then E is measurable.

So, let $E \subset X$ closed.
Let $B \subset X$. We must show
$$\mu(B) = \mu(B \cap E) + \mu(B \setminus E)$$
suffices to show "$\geq$".
May assume
$$\mu(B) < \infty.$$
Partition X into E, and rings around E, using the distance function



$\inf_{y \in E} \{d(x,y)\}$

$R_1 = \{x \in X$ s.t. $d(x,E) \geq 1\}$

$j > 1$: $R_j = \{x \in X$ s.t. $\frac{1}{j} \leq d(x,E) < \frac{1}{j-1}\}$.

Notice (because E is closed!)
that $x \in E$ iff $d(x, E) = 0$

Therefore $\quad E \cup \bigcup_{j=1}^{\infty} R_j \quad$ partitions X (into disjoint subsets).

Notice each ring R_j is separated from E_j (by distance $\frac{1}{j}$)
and if $(j-k) \geq 2$, then R_j is separated from R_k by $\frac{1}{k-1} - \frac{1}{j} > 0$
$j \geq k+2$ (reverse Δ ineq.)

Therefore, by the metric outer measure property, (and containment ineq)

$$(1) \quad \mu(B \setminus E) \geq \mu\Big(B \cap \bigcup_{\substack{j \text{ odd} \\ j \leq N}} R_j\Big) = \sum_{\substack{j \text{ odd} \\ j \leq N}} \mu(B \cap R_j)$$

$$(2) \quad \mu(B \setminus E) \geq \mu\Big(B \cap \bigcup_{\substack{j \text{ even} \\ j \leq N}} R_j\Big) = \sum_{\substack{j \text{ even} \\ j \leq N}} \mu(B \cap R_j)$$

$$(1),(2) \Rightarrow \boxed{\sum_{j=1}^{\infty} \mu(B \cap R_j) \leq 2\mu(B \setminus E) \leq 2\mu(B) < \infty.}$$

Finally,

separated by $\frac{1}{N}$

$$\mu(B) \geq \mu\Big(B \cap \big(E \cup \bigcup_{j=1}^{N} R_j\big)\Big) = \mu(B \cap E) + \mu\Big(B \cap \bigcup_{j=1}^{N} R_j\Big)$$

$\underline{\underline{\text{if}}}$ $\boxed{\lim_{N \to \infty} \mu\Big(B \cap \bigcup_{j=1}^{N} R_j\Big) \geq \mu(B \cap E^c)}$, we deduce the desired inequality.

$$\text{But, } \mu(B \cap E^c) = \mu\Big[\Big(B \cap \bigcup_{j=1}^{N} R_j\Big) \cup \Big(B \cap \bigcup_{j=N+1}^{\infty} R_j\Big)\Big]$$

$$\text{so} \quad \mu(B \cap E^c) \leq \mu\Big(B \cap \bigcup_{j=1}^{N} R_j\Big) + \mu\Big(B \cap \bigcup_{j=N+1}^{\infty} R_j\Big)$$

$$\text{so} \quad \mu(B \cap E^c) \leq \mu\Big(B \cap \bigcup_{j=1}^{N} R_j\Big) + \underbrace{\sum_{j=N+1}^{\infty} \mu(B \cap R_j)}_{} \quad \text{Subadditivity}$$

$$\lim_{N \to \infty}: \quad \mu(B \cap E^c) \leq \lim_{N \to \infty} \mu\Big(B \cap \bigcup_{j=1}^{N} R_j\Big) + 0$$

$\to 0$ as $N \to \infty$ since it's tail of conv. series.

More discussion about Hausdorff measure:

Scaling and Hausdorff measure

If (X,d) is a normed vector space, $A \subset X$, then for $t>0$, $\vec{b} \in X$

$$tA + \vec{b} := \{ x = ta + b \text{ s.t. } a \in A \}.$$

Theorem Let $\alpha > 0$. Then

$$\mu_\alpha(tA+b) = t^\alpha \mu_\alpha(A)$$

proof: Every cover of A by sets $\{F_k\}$ of diam $< \varepsilon$ corresponds to a cover of $tA+b$ by sets $\{G_k\}$ of diam $< t\varepsilon$. $\left(\begin{matrix} G_k = tF_k + b \\ F_k = \frac{1}{t}(G_k - b) \end{matrix} \right)$

And, $$\sum_k \operatorname{diam}(G_k)^\alpha = \sum_k t^\alpha \operatorname{diam}(F_k)^\alpha = t^\alpha \sum_k \operatorname{diam}(F_k)^\alpha$$

Thus $$\begin{array}{ccc} \mu_\alpha^{t\varepsilon}(tA+b) & = & t^\alpha \mu_\alpha^{\varepsilon}(A) \\ \Big\downarrow{\scriptstyle \varepsilon \to 0} & & \Big\downarrow{\scriptstyle \varepsilon \to 0} \\ \mu_\alpha(tA+b) & = & t^\alpha \mu_\alpha(A) \end{array}$$ ∎

example: Let C be the Cantor set, then $C = (\frac{1}{3}C) \cup (\frac{1}{3}C + \frac{2}{3})$ is a disjoint separated union

so $$\mu_\alpha(C) = \left(\tfrac{1}{3}\right)^\alpha \cdot 2 \cdot \mu_\alpha(C)$$

So the only α for which $0 < \mu_\alpha(C) < \infty$ could be possible is

$$2\left(\tfrac{1}{3}\right)^\alpha = 1 \quad ; \quad 2 = 3^\alpha \qquad \alpha = \frac{\ln 2}{\ln 3}$$

(This α is called the <u>scaling</u> dimension of C, & it turns out to be the Hausdorff dimension in this case - scaling dimension is always $\geq$ Hausdorff dimension)

example: Scaling dimension of Sierpinski triangle?

Finite covers for closed sets:

Let $A \subset X$, (X,d) metric

Let A compact, $\alpha > 0$

Define $\tilde{\mu}_\alpha^\varepsilon(A) = \inf\limits_{\substack{A \subset \bigcup_{k=1}^N F_k \\ F_k \text{ closed} \\ \operatorname{diam}(F_k) \le \varepsilon}} \left\{ \sum_{k=1}^N \operatorname{diam}(F_k)^\alpha \right\}$ i.e. only finite covers

Then $\mu_\alpha^{\varepsilon/2}(A) \geq \tilde{\mu}_\alpha^\varepsilon(A) \geq \mu_\alpha^\varepsilon(A)$, so also $\tilde{\mu}_\alpha^\varepsilon(A) \nearrow \mu_\alpha(A)$

$\mu_\alpha^{\varepsilon/2}(A) \xrightarrow{\varepsilon \to 0} \mu_\alpha(A)$, $\quad \mu_\alpha^\varepsilon(A) \xrightarrow{\varepsilon \to 0} \mu_\alpha(A)$

clearly by set containment

Let $A \subset \bigcup_{k=1}^\infty F_k$, F_k closed, $\operatorname{diam}(F_k) \le \varepsilon/2$.

Let $e > 0$. Pick $F_k \subset \mathcal{O}_k$ open, $\mathcal{O}_k = N_{\delta_k}(F_k)$, open nbhd, i.e. $\{x \in X \text{ s.t. } d(x, F_k) < \delta_k\}$

Note, $\operatorname{diam}(\mathcal{O}_k) \le 2\delta_k + \operatorname{diam}(F_k)$

Pick δ_k small s.t.
$\operatorname{diam}(\mathcal{O}_k) \le \varepsilon$
$(\operatorname{diam} \mathcal{O}_k)^\alpha \le (1+e)(\operatorname{diam}(F_k))^\alpha$

$$\boxed{\Rightarrow \sum_k \operatorname{diam}(\mathcal{O}_k)^\alpha \le (1+e) \sum \operatorname{diam}(F_k)^\alpha}$$

furthermore, since A is compact, $\exists$ finite subcover $\mathcal{O}_1, \ldots \mathcal{O}_N$, (so also $\overline{\mathcal{O}}_1, \ldots \overline{\mathcal{O}}_N$)
and
$$\sum_{k=1}^N \left(\operatorname{diam}(\overline{\mathcal{O}_k})\right)^\alpha \le (1+e) \sum \operatorname{diam}(F_k)^\alpha$$

$\Rightarrow \tilde{\mu}_\alpha^\varepsilon(A) \le (1+e)\mu_\alpha^{\varepsilon/2}(A)$. Let $e \searrow 0$ ∎