Math 5210
Tuesday April 7
§14.2 Strichartz .... recall the setup:

Let X be any space. An <u>outer measure μ</u> is a function from the subsets of X, to $[0,\infty]$, satisfying

1. $\mu(A)\in[0,\infty] \quad \forall A\subset X$
2. $\mu(\emptyset)=0$
3. $A\subset B \Rightarrow \mu(A)\le\mu(B)$
4. $\mu(\bigcup_n A_n) \le \sum_n \mu(A_n)$

If (X,d) is a metric space, then μ is a <u>metric outer measure</u> if it satisfies 1-4, and

5. Whenever A, B are separated by some $\delta>0$, $\quad (d(a,b)\ge\delta \ \ \forall a\in A, b\in B)$
   then $\mu(A\cup B)=\mu(A)+\mu(B)$.

If μ is an outer measure on X, then $E\subset X$ is <u>measurable</u> (or <u>satisfies the splitting condition</u> iff $\forall A\subset X$,

$$\mu(A)=\mu(A\cap E)+\mu(A\setminus E).$$

<u>Example 1</u> Let X be any space, define $\mu_0(A) = \#$ of elts in A.
μ_0 is called the <u>counting measure</u>. Every subset of X is measurable wrt μ_0

<u>Example 2</u> Lebesgue outer measure on $\mathbb{R}^n$.
For a coord. box $I=I_1\times I_2\times\ldots\times I_n\subset\mathbb{R}^n$ (each I_j a bounded interval)
define the volume of I by $|I|=\ell(I_1)\ell(I_2)\cdots\ell(I_n)$.

Let $A\subset\mathbb{R}^n$
Define

$$m^*(A) = \inf_{\substack{A\subset \bigcup I_n \\ \uparrow \\ \text{open boxes}}} \Big\{\sum_n |I_n|\Big\} = \inf_{\substack{A\subset \bigcup I_n \\ \uparrow \\ \text{closed boxes}}} \Big\{\sum_n |I_n|\Big\}$$

Check that m^* is a metric outer measure:

check, as we did in $\mathbb{R}$:

$\ge$: If $A\subset\bigcup_n I_n$, I_n open, then $A\subset\bigcup_n \bar{I}_n$.

$\le$: If $A\subset\bigcup_n I_n$, I_n closed, pick $I_n\subset J_n$ open, $|J_n|\le|I_n|+\varepsilon/2^n$
$\Rightarrow$ LHS $\le$ RHS $+\varepsilon$ $\forall\varepsilon$. ■

Example 3 α-dim'l Hausdorff measure.

Let (X, d) be a metric space

let $\alpha \in [0, \infty)$.

For $F \subset X$, $\operatorname{diam}(F) := \sup\limits_{x,y \in F} \{d(x,y)\}$

For $\varepsilon > 0$, $A \subset X$

$$\mu_\alpha^\varepsilon(A) := \inf_{\substack{A \subset \bigcup_n F_n \\ \operatorname{diam}(F_n) \leq \varepsilon \\ F_n \text{ closed}}} \left\{ \sum_n (\operatorname{diam}(F_n))^\alpha \right\}$$

$\longleftarrow$ units: is a sum of $(\text{lengths})^\alpha$

$\alpha = 1 \rightarrow$ lengths

$\alpha = 2 \rightarrow$ (multiple of) area

etc.

$$\mu_\alpha(A) := \lim_{\varepsilon \to 0} \mu_\alpha^\varepsilon(A) \in [0, \infty], \quad \text{since } \{\mu_\alpha^\varepsilon(A)\} \nearrow \text{ in } \varepsilon.$$

α-dimensional Hausdorff measure

(If $(X, d) = (\mathbb{R}^n, |\cdot|)$, then for $\alpha = k \in \mathbb{N}$, $1 \leq k \leq n$, then up to normalizing constants, μ_α generalizes k-dimensional area in $\mathbb{R}^n$)

You check in HW that μ_α is an outer measure.

We check condition (S) ~ metric outer measure, on the next page

5) Let A, B be separated by $\delta > 0$. Then

$$\mu_\alpha(A \cup B) = \mu_\alpha(A) + \mu_\alpha(B)$$

proof: Suffices to prove $\geq$, since $\leq$ is just subadditivity (4).

May assume

$$\mu_\alpha(A \cup B) < \infty.$$

Let $\varepsilon < \delta$. Let $\rho > 0$.

Pick a cover

$$A \cup B \subset \bigcup_k F_k \quad , \quad \mathrm{diam}(F_k) \leq \varepsilon$$

$$* \qquad \sum_k \mathrm{diam}(F_k)^\alpha \leq \mu_\alpha^\varepsilon(A \cup B) + \rho.$$

Then

$$\bigcup_k F_k = \underbrace{\bigcup_{\substack{k \text{ s.t.} \\ F_k \cap A \neq \emptyset}} F_k}_{\text{covers } A} \underset{\substack{\uparrow \\ \text{disjoint} \\ \text{union}}}{\cup} \underbrace{\bigcup_{\substack{k \text{ s.t.} \\ F_k \cap A = \emptyset}} F_k}_{\text{covers } B}$$

$$** \quad \Rightarrow \sum_k \mathrm{diam}(F_k)^\alpha \geq \mu_\alpha^\varepsilon(A) + \mu_\alpha^\varepsilon(B)$$

$$*, ** \Rightarrow \mu_\alpha^\varepsilon(A \cup B) + \rho \geq \mu_\alpha^\varepsilon(A) + \mu_\alpha^\varepsilon(B)$$

ρ arbitrary $\Rightarrow \mu_\alpha^\varepsilon(A \cup B) \geq \mu_\alpha^\varepsilon(A) + \mu_\alpha^\varepsilon(B)$

Let $\varepsilon \to 0$ ■

Hausdorff dimension

Theorem Let (X,d) metric space, $A \subset X$.
Then $\exists!\ \alpha \in [0,\infty]$, called the Hausdorff dimension of A s.t.

$$\gamma < \alpha \Rightarrow \mu_\gamma(A) = +\infty$$
$$\alpha < \beta \Rightarrow \mu_\beta(A) = 0$$

($\mu_\alpha(A)$ can be $0, \neq 0, +\infty$).

proof (this feels like the "radius of convergence" proof!).
Let $\varepsilon > 0$.
Consider any $0 \leq \gamma < \beta < \infty$.

Let $A \subset \bigcup_k F_k$
$\operatorname{diam}(F_k) \leq \varepsilon$

Then $$\sum_k \operatorname{diam}(F_k)^\beta \leq \sum_k (\operatorname{diam} F_k)^\gamma \varepsilon^{\beta-\gamma} = \varepsilon^{\beta-\gamma} \sum_k \operatorname{diam}(F_k)^\gamma$$

$$\Rightarrow \quad \mu_\beta^\varepsilon(A) \leq \varepsilon^{\beta-\gamma} \mu_\gamma^\varepsilon(A)$$

$$\begin{tikzcd} \mu_\beta^\varepsilon(A) \arrow[d, "\varepsilon\to 0"] & \mu_\gamma^\varepsilon(A) \arrow[d, "\varepsilon\to 0"] \\ \mu_\beta(A) & \mu_\gamma(A) \end{tikzcd}$$

$\gamma < \beta$: Thus $\mu_\gamma(A) < \infty \Rightarrow \mu_\beta(A) = 0$
and $\mu_\beta(A) > 0 \Rightarrow \mu_\gamma(A) = \infty$.

Thus $\{\gamma \in [0,\infty) \text{ s.t. } \mu_\gamma(A) = +\infty\}$
is either empty ← set $\alpha = 0$. Then $\beta > 0 \Rightarrow \mu_\beta(A) = 0$.
or it's $\{0\}$ ←
or it's an interval $I_1 = [0,\alpha)$
or $[0,\alpha]$

Set $\alpha := \sup\{x \in I_1\} \in [0,\infty]$.

Then $\gamma < \alpha \Rightarrow \mu_\gamma(A) = +\infty$
$\beta > \alpha \Rightarrow \mu_\beta(A) = 0$

∎

Example: The Cantor set has Hausdorff dim $\frac{\ln 2}{\ln 3}$