Math 5210
Monday April 6.

- Finish Friday's notes!

- Re cantor set $C = \bigcap_k C_k$

for the contraction $\tilde{f} : (\mathcal{K}, d_H) \to (\mathcal{K}, d_H)$

$$\tilde{f}(K) = f_1(K) \cup f_2(K)$$

$$f_1(x) = \tfrac{1}{3}x \ ; \quad f_2(x) = \tfrac{1}{3}x + \tfrac{2}{3}$$

we saw $\tilde{f}(C_k) = C_{k+1}$

we claimed that C is the fixed point of $\tilde{f}$;
this is because on one hand

$$\{\tilde{f}^k(C_0)\} \to \text{fixed pt (in } d_H).$$

but on the other hand,

because $C \subset C_k$
and $C_k \subset N_{1/3^k}(C)$ since each subinterval in C_k contains Cantor set pts, e.g. the endpoints!

$$d_H(\tilde{f}^k(C_0), C) = d_H(C_k, C) \le \frac{1}{3^k} \to 0$$

continue Friday notes p.2; ternary expansions

extension of $g : C \twoheadrightarrow [0,1]$ to a cont. fn from $[0,1] \to [0,1]$.

let $x \in [0,1]$,

$$x = \sum_{j=1}^{\infty} a_j 3^{-j}. \quad \text{If } a_j \in \{0,2\} \ \forall j, \ g(x) := \sum_{j=1}^{\infty} \left(\frac{a_j}{2}\right) 2^{-j} \quad \text{(as before)}$$

If $a_j \notin \{0,2\} \ \forall j$ let N be the smallest index with $a_N = 1$.

then $g(x) := \sum_{j=1}^{N-1} \frac{a_j}{2} 2^{-j} + \frac{a_N}{2^N}$

The graph of g is a fractal
because

$$g\left(\tfrac{1}{3}x\right) = \tfrac{1}{2} g(x)$$

$$g\left(\tfrac{1}{3}x + \tfrac{2}{3}\right) = \tfrac{1}{2} g(x) + \tfrac{1}{2}$$

$$g(x) = \tfrac{1}{2}, \quad 1/3 < x < 2/3$$

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Recap (and articulation to Strichartz)

We defined outer Lebesgue measure, $m^*(A)$, for arbitrary subsets of $A \subset \mathbb{R}$

We defined a measurable set E to be one for which

$$m^*(A) = m^*(A \cap E) + m^*(A \sim E)$$

holds $\forall A \subset \mathbb{R}$.

Strichartz calls this the splitting condition for E, and uses different letters for the sets, p. 644.

We proved the measurable sets form a Boolean algebra,

Strichartz calls a Boolean algebra a field, and he calls a σ-algebra a σ-field.

Then we showed the measurable sets are actually a σ-algebra, denoted by $\mathcal{M}$

Finally, we showed that the measurable sets include the Borel σ-algebra (:= the intersection of all σ-algebras containing the open sets, hence the smallest such σ-algebra), by showing every open subset of $\mathbb{R}$ is measurable.

Restricting m^* to $\mathcal{M}$, we denote it by m, and call it a measure:

Definition: A measure μ on a σ-algebra $\mathcal{F}$ is a function $\mu : \mathcal{F} \to [0, \infty]$ s.t. $\mu(\emptyset) = 0$ and

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

whenever $\{E_j\} \subset \mathcal{F}$ are disjoint.

(which implies all the other nice properties about containment, increasing unions, decreasing intersections, etc.)

We can repeat this process with other ways of "measuring" sets – see pages 6, 5 Friday notes, and to be continued!

Today: check that the examples on page 5 Friday are metric outer measures, if we have time. (well, except for Hausdorff measure, which is HW!)