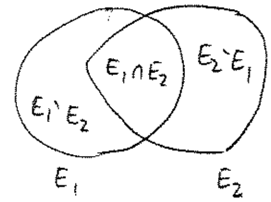
Math 5210
Friday April 3

HW for Friday April 10
Strichartz p 654 # 2, 3, 5, 8
Royden p. 69-70 # 22, 24, 25
Class exercise : Find the Hausdorff dimension of Sierpinski's triangle.

1

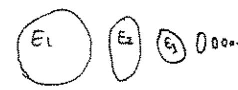
- The algebra of ~~measuring~~ measurable sets
(All sets below are assumed msble)



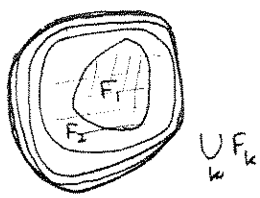

- $m(E_1 \setminus E_2) = m(E_1) - m(E_1 \cap E_2)$

- $m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$

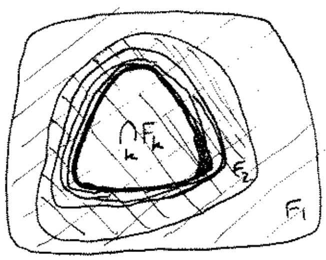
- $\{E_k\}$ disjoint $\Rightarrow m(\cup E_k) = \sum_{k=1}^{\infty} m(E_k) \quad (= \lim_{n\to\infty} S_n)$




(1) • $\{F_k\} \nearrow \ (F_k \subset F_{k+1}\ \forall k) \Rightarrow m(\bigcup_k F_k) = \lim_{k\to\infty} m(F_k)$




(2) • $\{F_k\} \searrow \ (F_k \supset F_{k+1}\ \forall k)$ and $m(F_1) < \infty \Rightarrow m(\bigcap_k F_k) = \lim_{k\to\infty} m(F_k)$




We proved (1) on Wednesday ( $\bigcup_k F_k = F_1 \cup (F_2 \setminus F_1) \cup (F_3 \setminus F_2) \cup \dots$ is a disjoint union, so
$m(\bigcup_k F_k) = m(F_1) + (m(F_2) - m(F_1)) + (m(F_3) - m(F_2)) + \cdots$

partial sum of positive ($\geq 0$) term series on the right is $m(F_n)$ ) ■

<u>Proof of 2</u> $F_1$ is a ctble union of the disjoint sets :

$$F_1 = (\bigcap_k F_k) \cup (F_1 \setminus F_2) \cup (F_2 \setminus F_3) \cup \dots$$

$$\Rightarrow \quad m(F_1) = m(\bigcap_k F_k) + (m(F_1) - m(F_2)) + (m(F_2) - m(F_3)) + \dots (m(F_n) - m(F_{n+1})) + \cdots$$

partial sums $= m(F_1) - m(F_n)$

$$\Rightarrow \quad m(F_1) = m(\bigcap_k F_k) + m(F_1) - \lim_{n\to\infty} m(F_n)$$

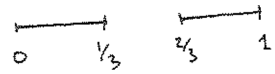
cancel if $m(F_1) < \infty$. ■

- The Cantor set and Cantor ternary function.
  - Please cross out the statement in Wed's notes, page 4, that the $f:[0,1]\to C$ was a bijection, it isn't quite one!

Actually, the sketch there has some other errors as well.

$C_0 = [0,1]$


$C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$




$C_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$

(delete middle thirds)

$C = \bigcap_k C_k$ is closed

It is also the fixed compact set of the contraction mapping on ~~sets~~ compact subsets of $\mathbb{R}$ (contraction with respect to Hausdorff distance)

of $\mathcal{F}(K) = f_1(K) \cup f_2(K)$

$$f_1(x) = \tfrac{1}{3}x, \quad f_2(x) = \tfrac{1}{3}x + \tfrac{2}{3}$$

- $f_1, f_2$ are contractions of $\mathbb{R}$
  so $\mathcal{F}$ is a contraction of $(\mathcal{K}, d_H)$
- $\mathcal{F}(C_k) = C_{k+1}$, so $\{C_k\} \xrightarrow{d_H}$ the unique fixed point

and $\bigcap_k C_k$ is <u>a</u> fixed point,

so $C$ is the fixed point!

& $m(C) \le m(C_k) = (\frac{2}{3})^k \to 0$, so $m(C) = 0$.

<u>ternary expansions</u>

$x \in [0,1] \Rightarrow x$ has an almost unique ternary expansion

$$x = \sum_{j=1}^{\infty} a_j 3^{-j} \qquad a_j \in \{0,1,2\}$$

write $x = 0.a_1a_2a_3\cdots$

Note, $\frac{1}{3} = 0.02\bar{2}\ldots = \sum_{j=2}^{\infty} 2\cdot 3^{-j} = \frac{2}{9}\left(\frac{1}{1-\frac{1}{3}}\right) = \frac{2}{9}\frac{3}{2}$ ✓

$= 0.10\bar{0}\ldots$

$\frac{2}{3} = 0.2\bar{0}\ldots$

$= 0.12\bar{2}\ldots$

Looking at the contraction mappings $f_1, f_2$

we see that if

$$x = \sum_{j=1}^{\infty} a_j 3^{-j} = .a_1 a_2 a_3 \cdots$$

then $f_1(x) = \frac{1}{3}x = \sum_{j=1}^{\infty} a_j 3^{-j+1} = .0 a_1 a_2 a_3 \cdots$

$f_2(x) = \frac{1}{3}x + \frac{2}{3} = .2 a_1 a_2 \cdots$

Notice, $x \notin C_1$

iff $x \in (1/3, 2/3)$

iff every ternary expansion of $x$ has $a_1 = 1$

and

$x \in C_1$ iff has a ternary expansion with $a_1 = 0$ or $a_1 = 2$.

$x \in C_1$ iff $x$ has a ternary expansion with $a_1 \neq 1$

Since $\mathcal{F}(C_1) = f_1(C_1) \cup f_2(C_1) = C_2$

$x \in C_2$ iff $x$ has a ternary expansion with $a_1, a_2 \neq 1$.

By induction,

$x \in C_k$ iff $x$ has a ternary expansion with all of $a_1, a_2, \ldots a_k \neq 1$

<u>Conclusion</u>

$x \in C$ <u>iff</u> $x$ has a ternary expansion consisting of only zeroes & 2's.

(and $x$ has exactly one such expansion).

Cantor ternary function

$$g : C \twoheadrightarrow [0,1]$$

$$g\left( \sum_{j=1}^{\infty} a_j 3^{-j} \right) := \sum_{j=1}^{\infty} \left(\frac{a_j}{2}\right) 2^{-j} .$$

$a_j \in \{0,2\}$

Since every $y \in [0,1]$ has a binary expansion

$$y = \sum_{j=1}^{\infty} b_j 2^{-j} \qquad b_j \in \{0,1\}$$

we see that for this $y$,

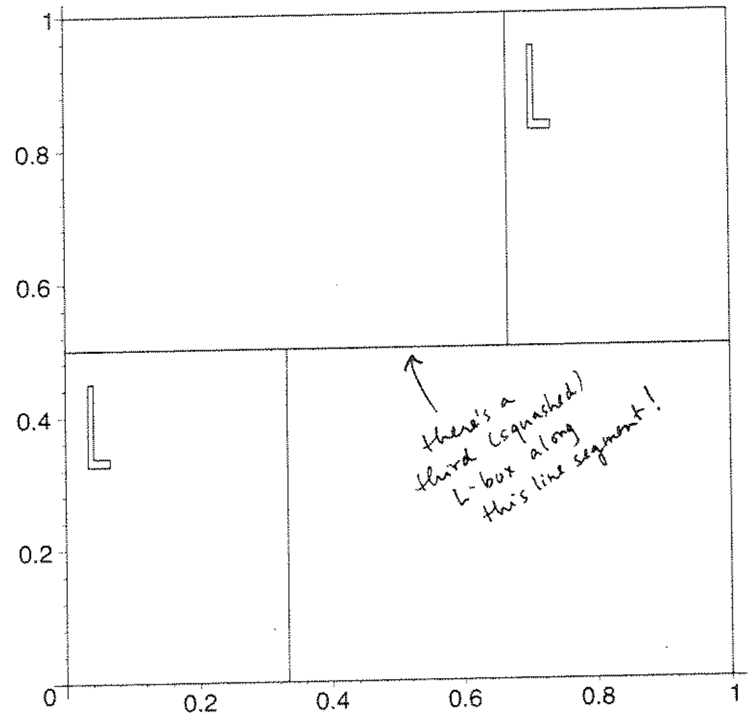
$$g\left( \sum_{j=1}^{\infty} (2b_j) 3^{-j} \right) = y$$

so $g$ is onto.

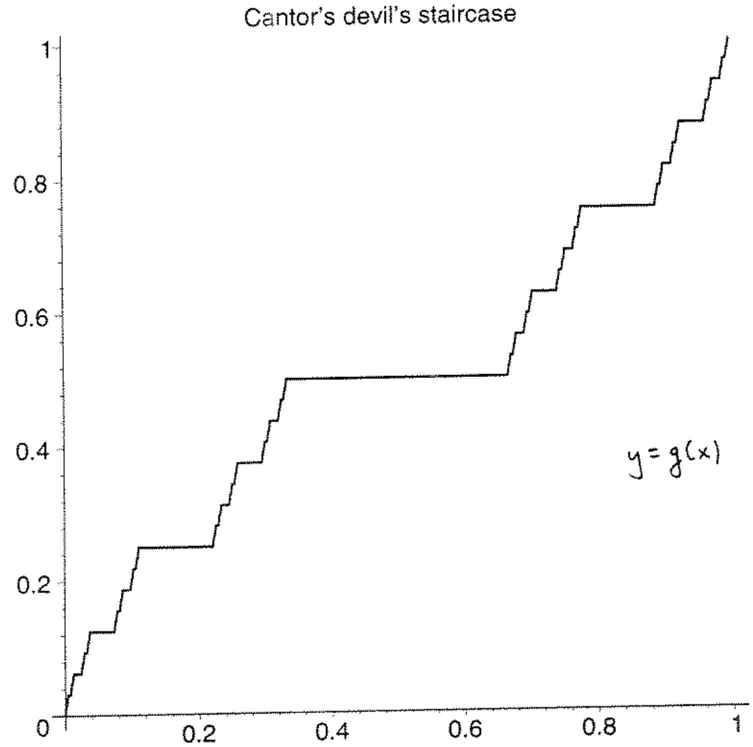
(We could have made $g$ a bijection with $(0,1]$ if we'd deleted intervals of the form $(a,b]$.)

In fact, $g$ extends to a continuous function $g : [0,1] \to [0,1]$, which is constant on each deleted interval, and the graph of $g$ is also a fractal, the Cantor staircase function.

fractal template




Cantor's devil's staircase



Other measures ( $\exists$ lots more too! )

- Lebesgue outer measure on $\mathbb{R}^n$.

  replace intervals with coordinate boxes $I = \underset{\text{intervals}}{I_1} \times I_2 \times \ldots \times I_n \subset \mathbb{R}^n$

  $= \{x \in \mathbb{R}^n \text{ s.t. } x_1 \in I_1, x_2 \in I_2, \ldots\}$

  $|I| := \ell(I_1)\,\ell(I_2)\cdots\ell(I_n)$

  $A \subset \mathbb{R}^n$

  $$m^*(A) := \inf_{\substack{A \subset \bigcup_n I_n \\ \uparrow \\ \text{open boxes}}} \sum_n |I_n|$$

  reproduces "volume"

- Counting measure (on any set)

  $\mu_0^*(A) = \#$ of elts of $A \in \{0,1,2,\ldots\} \cup \{\infty\}$.

- $\alpha$-dimensional Hausdorff measure. [we will spend some time on this].

  $\alpha \in [0, \infty)$

  $(X, d)$ metric space

  for $F \subset X$, $\operatorname{diam}(F) := \sup_{x,y \in F} \{d(x,y)\}$

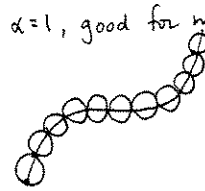
  For $A \subset X$

  consider countable covers $A \subset \bigcup F_n$, $F_n$ closed.

  For $\varepsilon > 0$

  $$\mu_\alpha^\varepsilon(A) := \inf_{\substack{A \subset \bigcup F_n,\ F_n \text{ closed} \\ \operatorname{diam}(F_n) \le \varepsilon\ \forall n.}} \sum_n \operatorname{diam}(F_n)^\alpha$$

  e.g. $\alpha = 1$, good for measuring length of a curve.



  as $\varepsilon \searrow 0$, $\mu_\alpha^\varepsilon(A) \nearrow$ because there are fewer admissable covers

  $$\boxed{\mu_\alpha(A) := \lim_{\varepsilon \to 0^+} \mu_\alpha^\varepsilon(A) \quad \in [0, \infty].}$$

The examples on page 5 are all metric outer measures. They satisfy

1. $\mu(A) \in [0, \infty] \quad \forall A \subset X$
2. $\mu(\emptyset) = 0$
3. $\mu(\bigcup_n A_n) \le \sum_n \mu(A_n)$
4. $A \subset B \Rightarrow \mu(A) \le \mu(B)$

"outer measure" axioms

5. If $A$ and $B$ are separated by $\delta > 0$, i.e. $d(a,b) \geq \delta \quad \forall a \in A, b \in B$, then $\mu(A \cup B) = \mu(A) + \mu(B)$

additional axiom for "metric outer measure"

We'll show next week that metric outer measures always yield a $\sigma$-algebra of measurable sets (for the particular outer measure), which always contains the Borel sets as a sub-sigma algebra.

When we restrict such metric outer measures to the measurable subsets we call them measures.

A measure $\mu$ on a $\sigma$-algebra $\mathcal{A}$ is a non-negative function $\mu : \mathcal{A} \to [0, \infty]$, $\mu(\emptyset) = 0$, which satisfies countable additivity:

If $\{A_k\} \subset \mathcal{A}$ are disjoint, then $\mu(\bigcup_k A_k) = \sum_{k=1}^{\infty} \mu(A_k)$

(Not all measures arise from metric outer measures)