Course theme:

metric spaces! "Solutions" to problems in math & science are often functions, and these functions are often elements of metric spaces like $C([a,b],X)$ or $L^p(X,\mu)$. This course has been an introduction to these spaces, and to the analysis techniques and theorems which let you find and approximate such "solutions."

Math 5210
Wednesday April 29.
Last class day!

Final exam is Wed. May 6, 1-3 p.m., in our classroom.
The optional takehome problems are due at the start of the final exam, or before. Do not consult other people on these problems, I want them to be your own effort.

- Discuss Tuesday notes about the key result: $L^p(X,\mu)$ is complete.
- Review course:

  Final exam overview: There may be some mandatory problems, as on the midterm, but you will also have significant choice in demonstrating that you understand and can explain important ideas from this course.... last time I taught this course I let students choose 6 out of 10 problems. Expect measure theory and the Lebesgue integral to be well-represented. In general, study definitions, theorems, examples.

<u>General Metric Spaces</u> Strichartz 2.1-2.3, 3.1, 9.1-9.2, class notes

metric space, convergent and Cauchy sequences.
complete metric spaces
$\mathbb{R}$ as a complete metric space (also $\mathbb{R}^n$)
  sups, infs of sets. limsups, liminfs, lims of sequences
  limits of sums, prods quotients of convergent sequences
normed vector spaces (Banach if complete)
inner product spaces (Hilbert if complete)
equivalent metrics, metric subspaces
open, closed subsets, interior, closure, complement, limit pt.
compactness (characterizations)
connectivity and path connectivity

<u>Continuous fcns between metric spaces</u> 9.3, 7.3-7.6
continuity def's
basic theorems:
  compositions
  sum, prods, inverses of cont. real-valued fcns.
  continuous images of compact or connected sets
    max-min thms if image is $\mathbb{R}$, also intermediate value thm.

* I like to ask these!

important harder thms
  contraction mapping principal
  uniform convergence
    uniform limits of cont. fcns are cont
    $C(X,Y)$ is complete if X is compact and Y is complete $\left(d(f,g)=\sup_{x\in X} d(f(x),g(x))\right)$
  continuous functions on compact sets are uniformly continuous
  equicontinuity and Arzela Ascoli Thm.

<u>Differentiability</u> 10.1

$f: X \to Y$ differentiable (affine approx)

$f: [a,b] \to X$ diff (tangent vector) (special case of affine approx).

* chain rule w operator norm

<u>Riemann integral</u> : best for continuous $f: [a,b] \to X_{\text{Banach}}$ class notes (also 6.1) 11.1, 7.3-7.5

* definition, existence, estimates
* FTC
* $\exists!$ for 1st order systems of DE's via contraction mapping principal.
* Exchange of limits and integrals (or sums), when convergence is uniform
* convolution, approximate identity estimates
  Weierstrass approx. thm (& Stone-Weierstrass theorem)

<u>Fourier series</u> (not on final exam)

<u>Measure theory</u> 14.1-14.2 Strichartz, 3 Royden

Lebesgue outer measure m^* (not on exam: Hausdorff outer measure, or Hausdorff distance)

general outer measure

metric outer measure

measurable sets of an outer measure (splitting cond. def.)

they form a <u>σ-algebra</u> (Royden def)

the measurable sets from a metric outer measure contain the <u>Borel sets</u>

measure space $(X, \mathcal{F}, \mu)$

properies of a measure μ from a measure space (increasing unions, decreasing $\cap$'s)

special case $(\mathbb{R}, \mathcal{M}, m)$; characterizations of measurable sets, interesting examples

<u>Lebesgue integration</u> Royden chapter 4, in general context $(X, \mathcal{F}, \mu)$ and special cases $(\mathbb{R}, \mathcal{M}, \mu)$, $(\mathbb{N}, 2^{\mathbb{N}}, \mu_c)$

* $\gtrsim 50\%$ of final exam $\gtrsim 40\%$)

$f: E \to \overline{\mathbb{R}}$ measurable

equivalent definitions

relation to continuity

properties, thms

$\boxed{\text{definitions, proofs of thms (and statements!), examples}}$

simple functions φ

special case: step fcns.

$\int_E \varphi \, d\mu$, φ simple

$\int_E f \, d\mu$, f bd, $\mu(E) < \infty$. $R\int_a^b f\,dx \ \exists \Rightarrow \int_a^b f(x)dx \ \exists$ and is equal to $R\int_a^b f(x)dx$.

bd conv. theorem

$\int_E f\,d\mu$, $f \geq 0$

Fatou's lemma; ~~bo~~ monotone convergence theorem

$\int_E f\,d\mu$, $f \in L^1(E)$.

Lebesgue dominated conv. thm.

also continuous parameter version. Comparison to thms for Riemann integral.

L^p spaces: p-norm, the facts that $L^p(X,\mu)$ is complete, $(C[a,b], \|\cdot\|_p)$ dense in $L^p([a,b],m)$

triangle ineq, Hölder ineq.