Math 5210
Tuesday April 28

- non-measurable set example, Royden 3.4.
- L^p spaces:

Let $1 \le p < \infty$.

$f \in L^p(X,\mu)$ iff
(i) $f: X \to \overline{\mathbb{R}}$ is measurable
(ii) $\int_X |f|^p \, d\mu < \infty$.

$p=1$: Exactly the Lebesgue integrable fns we considered over the past few lectures.

Speaking precisely, elements of L^p are equivalence classes of functions which agree a.e., so that we can consider $L^p(X,\mu)$ to be a metric space.
($d(f,g)=0$ iff $f=g$).

- $L^p(X,\mu)$ is a normed vector space:

<u>Vector space</u>: sums and scalar multiples of msble fns are msble.
Also,

$$\int_X |cf|^p \, d\mu = |c|^p \int_X |f|^p \, d\mu < \infty \quad \text{if } f \in L^p$$

$$\int_X |f+g|^p \, d\mu \le \int_X 2^p(|f|^p + |g|^p)\, d\mu < \infty \quad \text{if } f,g \in L^p.$$

$$|f+g| \le |f|+|g| \le 2\max(|f|,|g|)$$
$$\Rightarrow |f+g|^p \le 2^p \max(|f|^p,|g|^p)$$
$$\le 2^p(|f|^p+|g|^p)$$

<u>Norm</u>: $\|f\|_p := \left(\int_X |f|^p \, d\mu\right)^{1/p}$

$\|cf\|_p = |c|\,\|f\|_p$

$\|f+g\|_p \le \|f\|_p + \|g\|_p$ ← we essentially did this at start of term! (reviewed in Royden & page 2 today).

$\|f\|_p = 0$ iff $f=0$ a.e.

$\Leftarrow$ $f=0$ a.e. $\Rightarrow \int |f|^p \, d\mu = 0$ ∎

$\Rightarrow$: $\int_X |f|^p \, d\mu = 0 \Rightarrow |f|^p = 0$ a.e. This is HW p.86 #3!
Hint: Let $E_n = \{x \in X \text{ s.t. } |f(x)| \ge \frac{1}{n}\}$

Recall Minkowski & Δ ineq's. (We did this in HW!!)

starting point

geom-arith ineq: $\boxed{\begin{array}{l}\alpha,\beta\geq 0,\ 0<\lambda<1\\ \alpha^\lambda\beta^{1-\lambda}\leq\lambda\alpha+(1-\lambda)\beta\end{array}}$ (see old hw; text).

So,

$\begin{array}{l}p,q>1\\ \frac{1}{p}+\frac{1}{q}=1\\ \ \ \| \quad \|\\ \lambda \quad 1-\lambda\end{array}$: $|f(t)|\,|g(t)| \leq (|f(t)|^p)^{\frac{1}{p}}(|g(t)|^q)^{\frac{1}{q}}$

(1) $\leq \frac{1}{p}|f(t)|^p + \frac{1}{q}|g(t)|^q$

Cor1: Hölder Inequality : $f\in L^p$, $g\in L^q$ $(\frac{1}{p}+\frac{1}{q}=1,\ p,q>1)$ ← actually holds $p=1$, $q=\infty$ too

$\Rightarrow fg\in L^1$ and

$$\boxed{\int_X |fg|\,d\mu \leq \|f\|_p\|g\|_q}$$

proof: In case $\|f\|_p=\|g\|_q=1$ this follows by integrating (1)

Thus $\displaystyle\int_X \frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q}\,d\mu \leq 1$

$$\| $$

$\displaystyle\frac{1}{\|f\|_p}\frac{1}{\|g\|_q}\int_X |f||g|\,d\mu \leq 1$ ∎

Cor2 (Δ inequality, AKA Minkowski).

$\boxed{f,g\in L^p \Rightarrow \|f+g\|_p\leq\|f\|_p+\|g\|_p}$

Pf. $\displaystyle\int_X |f+g|^p\,d\mu \leq \int_X |f+g|^{p-1}(|f|+|g|)\,d\mu$

$\|f+g\|_p^p$

$$= \int_X |f|\,|f+g|^{p-1}\,d\mu + \int_X |g|\,|f+g|^{p-1}\,d\mu$$

$$\overset{\text{Hölder}}{\leq} \|f\|_p \underbrace{\|(f+g)^{p-1}\|_q}_{\|f+g\|_p^{p-1}} + \|g\|^p\|(f+g)^{p-1}\|_q \qquad q=\frac{p}{p-1}$$

$$= \|f+g\|_p^{p-1}(\|f\|_p+\|g\|_p)$$ ∎!

- $L^p(X,\mu)$ is a Banach space, i.e. it's a complete normed vector space

Lemma: Let V be a normed vector space. Then V is complete if and only if every absolutely summable series is summable

absolutely summable: $\sum_{k=1}^{\infty} v_k$ s.t. $\sum_{k=1}^{\infty} \|v_k\| < \infty$

summable: $\sum_{k=1}^{\infty} v_k$ s.t. $\exists\, v \in V$ s.t. $s_n = \sum_{k=1}^{n} v_k \to v$

proof of lemma:

$\Rightarrow$: Let V complete, $\sum v_k$ s.t. $\underbrace{\sum \|v_k\|}_{= L} < \infty$

This implies $\{s_n\}$ is Cauchy, since $\forall \varepsilon > 0\ \exists\, M$ s.t. $m, n \geq M \Rightarrow$

$$\sum_{m+1}^{n} \|v_k\| < \varepsilon$$

$$\Rightarrow \|s_n - s_m\| = \left\| \sum_{m+1}^{n} v_k \right\| < \varepsilon$$

Since V is complete, $\{s_n\} \to v \in V$. ∎

Slick direction:

$\Leftarrow$: Let every abs. sum. series be summable

Let $\{w_k\} \subset V$ be Cauchy.

$$\Rightarrow \exists\, m_j \text{ s.t. } k, \ell \geq m_j \Rightarrow \|w_k - w_\ell\| < \frac{1}{2^j}$$

Pick subsequence $\{w_{k_j}\}$ $(k_{j+1} > k_j)$, $k_j > m_j$

$$\Rightarrow \|\underbrace{w_{k_{j+1}} - w_{k_j}}_{:= v_j}\| < 1/2^j, \text{ so } \sum \|v_j\| < \infty$$

but $\underbrace{w_{k_{n+1}} = w_{k_1} + \sum_{j=1}^{n} v_j}$

$\downarrow$ $n \to \infty$, by hypothesis

w

$\Rightarrow \{w_k\}$ has convergent subsequence.

But in any metric space, a Cauchy seq. with a convergent subsequence, converges (to that limit)

$$(w - w_\ell = w - w_{k_j} + w_{k_j} - w_\ell)$$

$\Rightarrow V$ is complete

∎

Proof that $L^p(X,\mu)$ is complete $(1 \le p < \infty)$. ($p=\infty$ also true – like proof that uniformly Cauchy seq. of cont fcns converges in sup norm).

We use the lemma.

Let $\sum_{k=1}^{\infty} \|f_k\|_p = M < \infty$

We must show $\sum_{k=1}^{n} f_k \xrightarrow{L^p} f$, some $f \in L^p$.

proof define $g_n(x) = \sum_{k=1}^{n} |f_k(x)|$

$$\Delta \text{ ineq} \Rightarrow \|g_n\|_p \le \sum_{k=1}^{n} \|f_k\|_p \le M \Rightarrow \int_X |g_n(x)|^p \, d\mu \le M^p$$

$\{g_n(x)\} \uparrow$ in n, has limit $g(x)$

$$\text{Fatou} \Rightarrow \int_X |g(x)|^p \, d\mu \le M^p$$

$$\Rightarrow |g(x)| < \infty \text{ a.e.}$$

everywhere $|g(x)| < \infty$, the series $\sum_k |f_k(x)| < \infty$

$\Rightarrow \sum_k f_k(x)$ converges there, in $\mathbb{R}$, let $s(x) = \begin{cases} \sum_{k=1}^{\infty} f_k(x) & \text{if series conv} \\ 0 & \text{if not (measure zero set)} \end{cases}$

- Then $s(x)$ is measurable. [~~also set $f_k(x)=0$ on this bad set~~].
- $|s(x)| \le \sum_{k=1}^{\infty} |f_k(x)| = g(x)$ a.e.

  $\Rightarrow \int_X |s(x)|^p \, d\mu \le \int_X g(x)^p \, d\mu < \infty$, so $s \in L^p(X,\mu)$

- For $s_n(x) = \sum_{k=1}^{n} f_k(x)$, $\quad |s(x) - s_n(x)|^p \le |g(x) + g(x)|^p \le 2^p |g(x)|^p$ a.e.

  so $\{|s - s_n|^p\}$ is a sequence of integrable function converging a.e. to zero. They are dominated by the integrable fcn $2^p |g(x)|^p$.

so, $\text{LDCT} \Rightarrow \int_X |s - s_n|^p \, d\mu \to \int_X 0 \, d\mu = 0$

$\Rightarrow \|s - s_n\|_p \to 0$ ▮

Final 20 points of Final exam mitigation problems

(All of these problems, should you choose to do them, are due at the start of the in-class final exam.)

IV For the measure space $(\mathbb{R}, \mathcal{M}, m)$, show that the continuous functions
$\downarrow$ $[a,b]$

10 points $C([a,b])$ are dense in the Banach space $L^p([a,b],m)$, $1 \le p < \infty$.
Follow the following outline (btw, this is like Royden #15 p.90, except for L^p rather than L^1.)

IVa) Prove the lemma that if $X_1, X_2, X_3 \subset (X,d)$ a metric space,
and if X_1 is dense in X_2
and X_2 is dense in X_3,
then X_1 is dense in X_3

b) Use the lemma above, and show X_j is dense in X_{j+1}, for
$X_1 = C([a,b])$
$X_2 = \{$step fcns on $[a,b]\}$
$X_3 = \{$simple fcns on $[a,b]\}$
$X_4 = \{$bounded msble fcns on $[a,b]\}$
$X_5 = L^p([a,b])$.

V Prove that $L^p([a,b])$ is isometric to the abstract completion
10 points of $(C[a,b], \|\cdot\|_p)$ that we discussed earlier in the course. Do this by proving the following "uniqueness theorem" for metric space completions:

<u>Theorem</u> Let (X_1, d) be dense in (X,d), where (X,d) is complete

Then (X,d) is (isometric to) the abstract completion of (X_1,d).

Hints: denote the abstract completion by $(\overline{X}_1, \bar{d}) = \{[\{x_k\}], \bar{d}\}$
$\uparrow$
equiv. classes of Cauchy seq's.
$\bar{d}([\{x_k\}],[\{y_k\}]) = \lim_{k\to\infty} d(x_k,y_k)$.

Define $\varphi: \overline{X}_1 \to X$ by
$\varphi([\{x_k\}]) = \lim_{k\to\infty} x_k \in X$.
Show φ is well-defined, 1-1, onto, and an isometry (preserves distances.)