Math 5210
Monday
April 27

Final exam mitigation problems:
(Solo work)

- Finish Friday notes
- non-measurable set

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Tomorrow: L^p spaces discussion.

Wed: review.

I 10 points (2.46)

21. Let p be an integer greater than 1, and x a real number, $0 < x < 1$. Show that there is a sequence $\langle a_n \rangle$ of integers with $0 \le a_n < p$ such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

and that this sequence is unique except when x is of the form q/p^n, in which case there are exactly two such sequences. Show that, conversely, if $\langle a_n \rangle$ is any sequence of integers with $0 \le a_n < p$, the series

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

converges to a real number x with $0 \le x \le 1$.

If $p = 10$, this sequence is called the *decimal* expansion of x. For $p = 2$ it is called the *binary* expansion; and for $p = 3$, the *ternary* expansion.

36. The **Cantor ternary set** C consists of all those real numbers in $[0, 1]$ which have ternary expansion (cf. Problem 21) $\langle a_n \rangle$ for which a_n is never 1. (If x has two ternary expansions, we put x in the Cantor set if *one* of the expansions has no term equal to 1.) Show that C is a closed set, and that C is obtained by first removing the middle third $(\frac{1}{3}, \frac{2}{3})$ from $[0, 1]$, then removing the middle thirds $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ of the remaining intervals, and so on.

37. Show that the Cantor set can be put into a one-to-one correspondence with the interval $[0, 1]$.

46. Let x be a real number in $[0, 1]$ with the ternary expansion $\langle a_n \rangle$ (cf. Problem 21). Let $N = \infty$ if none of the a_n are 1, and otherwise let N be the smallest value of n such that $a_n = 1$. Let $b_n = \frac{1}{2}a_n$ for $n < N$ and $b_N = 1$. Show that

$$\sum_{n=1}^{N} \frac{b_n}{2^n}$$

is independent of the ternary expansion of x (if x has two expansions) and that the function f defined by setting

$$f(x) = \sum_{n=1}^{N} \frac{b_n}{2^n}$$

is a continuous, monotone function on the interval $[0, 1]$. Show that f is constant on each interval contained in the complement of the Cantor ternary set (Problem 36), and that f maps the Cantor ternary set *onto* the interval $[0, 1]$. (This function is called the *Cantor ternary function*.)

II 5 points (3.15, 3.16)

15. Show that if E is measurable and $E \subset P$, then $mE = 0$. [Hint: Let $E_i = E \dot{+} r_i$. Then $\langle E_i \rangle$ is a disjoint sequence of measurable sets and $mE_i = mE$. Thus $\sum mE_i = m \bigcup E_i \le m[0, 1]$.]

16. Show that, if A is any set with $m^*A > 0$, then there is a nonmeasurable set $E \subset A$. [Hint: If $A \subset (0, 1)$, let $E_i = A \cap P_i$. The measurability of E_i implies $mE_i = 0$, while $\sum m^*E_i \ge m^*A > 0$.]

24. Let f be measurable and B a Borel set. Then $f^{-1}[B]$ is a measurable set. [Hint: The class of sets for which $f^{-1}[E]$ is measurable is a σ-algebra.]

25. Show that if f is a meaurable real-valued function and g a continuous function defined on $(-\infty, \infty)$, then $g \circ f$ is measurable.

26. *Borel measurability.* A function f is said to be **Borel measurable** if for each α the set $\{x: f(x) > \alpha\}$ is a Borel set. Verify that Propositions 18 and 19 and Theorem 20 remain valid if we replace "measurable set" by "Borel set" and "(Lebesgue) measurable" by "Borel measurable." Every Borel measurable function is Lebesgue measurable. If f is Borel measurable, and B is a Borel set, then $f^{-1}[B]$ is a Borel set. If f and g are Borel measurable, so is $f \circ g$. If f is Borel measurable and g is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.

III 15 points (3.28)

27. How much of the preceding problem can be carried out if we replace the class $\mathcal{B}$ of Borel sets by an arbitrary σ-algebra $\mathcal{A}$ of sets?

28. Let f_1 be the Cantor ternary function (cf. Problem 2.46), and define f by $f(x) = f_1(x) + x$.

a. Show that f is a homeomorphism of $[0, 1]$ onto $[0, 2]$.

b. Show that f maps the Cantor set onto a set F of measure 1.

c. Let $g = f^{-1}$. Show that there is a measurable set A such that $g^{-1}[A]$ is not measurable.

d. Give an example of a continuous function g and a measurable function h such that $h \circ g$ is not measurable. Compare with Problems 25 and 26.

e. Show that there is a measurable set which is not a Borel set.

20 more points tomorrow (Tuesday)

Step 2 Let $\delta > 0$.

$\forall n \in \mathbb{N}$ use Step 1, to find a set

$$F^{1/n}_{M_n} = \{x \in \mathcal{D} \text{ s.t. } |f_k(x) - f(x)| < \tfrac{1}{n} \ \forall k \geq M_n\}$$

$$\mu(\mathcal{D} \setminus F^{1/n}_{M_n}) < \delta/2^n.$$

Originally April 14 notes

Let $E_\delta := \bigcap_n F^{1/n}_{M_n}$

- $\forall x \in E_\delta$, $n \in \mathbb{N}$. $\exists M_n$ s.t. $k \geq M_n \Rightarrow |f_k(x) - f(x)| < 1/n$

  so $\{f_k\} \to f$ uniformly on E_δ

in $\mathcal{D}$,

- $E_\delta^c = \bigcup_n (F^{1/n}_{M_n})^c \Rightarrow \mu(E_\delta^c) \leq \sum_{n=1}^{\infty} \delta/2^n = \delta$ ∎

Compositions of measurable fns may not be measurable! (See HW.)

$(\mathbb{R}, \mathcal{M}, m)$

You may appeal to §3.4, $\exists$ a non-measurable set for Lebesgue outer measure m^*

proof: Let $E \subset [0,1]$. For $y \in [0,1)$ define

$$E \dotplus y = \left\{x \in [0,1] \text{ s.t. } \begin{array}{l} x = e + y \text{ for } e \in E, \\ \text{or } x = e + y - 1 \text{ for } e \in E \end{array}\right\}$$

(you are translating E by y, mod 1).

Lemma: $E \in \mathcal{M} \Rightarrow E \dotplus y \in \mathcal{M}$ and $m(E \dotplus y) = m(E)$

proof Let $0 \leq y < 1$. $E_1 := E \cap [0, 1-y)$

$E_2 : E \cap [1-y, 1]$

$$\left.\begin{array}{l} E_1 \dotplus y = E_1 + y \in \mathcal{M} \\ E_2 \dotplus y = E_2 + y - 1 \in \mathcal{M} \end{array}\right\} \text{ so } E \dotplus y = (E_1 + y) \cup (E_2 + y - 1) \in \mathcal{M}$$

and $m(E \dotplus y) = m(E_1) + m(E_2) = m(E)$. ∎

Now consider the equivalence relation on $[0,1)$, $x \sim y$ iff $x - y \in \mathbb{Q}$ the rational #'s.
This is an equivalence relation, so partitions $[0,1)$ into equivalence classes,
$[0,1) = \bigcup_{a \in A} E_a$. By the axiom of choice $\exists$ a set P s.t. P contains
exactly one element from each equivalence class. This implies

$$[0,1) = \bigcup_{q \in \mathbb{Q} \cap [0,1)} P \dotplus q \qquad \text{disjoint union!!}$$

If P were measurable, we'd have $1 = m([0,1)) = \sum_{q \in \mathbb{Q} \cap [0,1)} m(P)$ fails! ~~$m(P) = 0$~~ ~~$m(P) > 0$~~. ∎