Math 5210
Wednesday April 22

- Integrals of non-negative (msble) fcns on arbitrary (msble) domains
  pages 3-4 Tuesday notes, then § 4.4 : general Lebesgue integrals

add property:

(vii) $A \subset E$, $\mu(A) = 0 \Rightarrow \int_A f\,d\mu = 0$

(e.g. $f(x) = \begin{cases} +\infty & x \in C \\ 0 & x \in \mathbb{R} \setminus C \end{cases}$ what is $\int_{\mathbb{R}} f\,dx$?)

Convergence thms:

Fatou's Lemma $(X, \mathcal{F}, \mu)$, $E \in \mathcal{F}$

$\{f_k\}, f : E \to [0, \infty]$ non-negative msble fcns

$\{f_k\} \to f$ a.e. on E

Then $\boxed{\int_E f\,d\mu \le \liminf_{n\to\infty} \int_E f_n\,d\mu}$

(this is half of being able to interchange the limit $n \to \infty$ with the integration).

Proof: Since $\int_E f\,d\mu := \sup_{\substack{0 \le g \le f \\ g \text{ bd, msble,} \\ \text{sppted on set} \\ \text{of finite meas.}}} \int_E g$

It suffices to pick any such g and prove

$$\int_E g\,d\mu \le \liminf_{n\to\infty} \int_E f_n\,d\mu$$

proof: Define $g_k = \min(f_k, g)$ $\quad (= \frac{1}{2}(f_k + g) - \frac{1}{2}|f_k - g|$ is msble$)$

Then BCT applies to $\{g_k\}$, and $\{g_k\} \to g$ a.e.

(if $\{f_k(x)\} \to f(x)$
then for $g(x) < f(x)$
$g_k(x) \equiv g(x)$ k large
For $g(x) = f(x)$,
then $\{f_k(x)\} \to f(x) = g(x)$)

$$\int_E f_k\,d\mu \ge \int_E g_k\,d\mu$$

$$\Rightarrow \liminf_{n\to\infty} \int_E f_n\,d\mu \ge \liminf_{n\to\infty} \int_E g_n\,d\mu \overset{BCT}{=} \lim_{n\to\infty} \int_E g_n\,d\mu = \int_E g\,d\mu \quad \blacksquare$$

Cor Monotone Convergence Theorem (MCT)

$(X, \mathcal{F}, \mu)$, $E \in \mathcal{F}$

$\{f_k\}, f : E \to [0, \infty]$ non-neg, meas. fns.

If $\{f_k\} \nearrow f$ a.e. (i.e. $\{f_k(x)\}$ increasing seq. a.e. x)

Then $\int_E f_k \, d\mu \to \int_E f \, d\mu$

since each $\int_E f_n \, d\mu \le \int_E f \, d\mu$.

proof: $$\int_E f \, d\mu \underset{\substack{\uparrow \\ \text{Fatou}}}{\le} \liminf_{n\to\infty} \int_E f_n \, d\mu \le \limsup_{n\to\infty} \int_E f_n \, d\mu \le \int_E f \, d\mu \qquad \blacksquare$$

§4.4 General Lebesgue integrable functions

$(X, \mathcal{F}, \mu)$, $E \in \mathcal{F}$

$f : E \to \overline{\mathbb{R}}$ measurable

Define $$f^+(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) \le 0 \end{cases} \qquad f^+(x) = \frac{f(x) + |f(x)|}{2} \text{ is msble}$$

$$f^-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) \ge 0 \end{cases} \qquad f^-(x) = \frac{|f(x)| - f(x)}{2}$$

So $f^+, f^- \ge 0$, $f = f^+ - f^-$

Def f is Lebesgue integrable on E ($f \in L^1(E)$) iff

$$\int_E f^+ \, d\mu < \infty \quad \text{and} \quad \int_E f^- \, d\mu < \infty.$$

In this case $$\int_E f \, d\mu := \int_E f^+ \, d\mu - \int_E f^- \, d\mu$$

as well, $$\underset{\substack{\uparrow \\ \text{why we say } f \in L^1(E).}}{\int_E |f| \, d\mu} = \int_E f^+ + f^- \, d\mu < \infty.$$

Example

$(\mathbb{R}, \mathcal{M}, m)$

Is $f(x) = \frac{1}{x}$ $(x \neq 0)$ $\in L^1([-1,1])$?

How about $f(x) = \frac{1}{x^{1/3}}$?

Lemma If $f \in L^1(E)$, $f = f^+ - f^-$ also is a difference $f = f_1 - f_2$ with $f_1, f_2 \geq 0$ msble, integrable

then $\int_E f \, d\mu = \int_E f_1 \, d\mu - \int_E f_2 \, d\mu$

proof

$$f^+ - f^- = f_1 - f_2$$

$$f^+ + f_2 = f_1 + f^-$$

$$\int_E f^+ + f_2 \, d\mu = \int_E f_1 + f^- \, d\mu$$

$$\int_E f^+ d\mu - \int_E f^- d\mu = \int_E f_1 \, d\mu - \int_E f_2 \, d\mu \qquad \blacksquare$$

Properties of Lebesgue Integrable functions : Let $f, g \in L^1(E)$.

(1) for $c \in \mathbb{R}$, $cf \in L^1(E)$, $\int_E cf = c \int_E f$

(2) $f + g \in L^1(E)$, $\int_E f + g = \int_E f + \int_E g$ $\leftarrow$ use $(f+g) = (f^+ + g^+) - (f^- + g^-)$ and Lemma!

(3) $f \leq g \Rightarrow \int_E f \, d\mu \leq \int_E g \, d\mu$ $\leftarrow$ use linearity (1), (2) & $g - f \geq 0$.

(4) $A \subset E$ msble subset $\Rightarrow f \in L^1(A)$, $\int_A f \, d\mu = \int_E f \, \mathbb{1}_A \, d\mu$ (pf: true for f^+, f^-).

(5) $A, B \subset E$ disjoint, msble $\Rightarrow \int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu$

(6) $A \subset E$, $\mu(A) = 0 \Rightarrow \int_A f \, d\mu = 0$

(7) $f = g$ a.e., $\Rightarrow \int_E f \, d\mu = \int_E g \, d\mu$

Lebesgue Dominated Convergence Theorem (LDCT).

$(X, \mathcal{F}, \mu)$
$\quad\ \ \downarrow$
$\quad\ \ E$

Let $g \in L^1(E)$, $g \geq 0$
Let $\{f_n\} : E \to \overline{\mathbb{R}}$ measurable, & dominated by g, i.e. $|f_n| \leq g \ \forall n$.
Let $\{f_n\} \to f$ a.e. on E (actually only need $|f_n| \leq g$ a.e. for each n)

Then $\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$.

Proof: $\int_E g \, d\mu < \infty$; $|f_n| \leq g \Rightarrow f_n^+, f_n^- \leq g \Rightarrow f_n \in L^1(E)$.
$\{f_n\} \to f$ a.e. $\Rightarrow f \leq g$ a.e. & msble
$\Rightarrow f \in L^1(E)$.

$g - f_n \geq 0$

Fatou $\Rightarrow \int g - f \, d\mu \leq \liminf_{n\to\infty} \int g - f_n = \int g - \limsup_{n\to\infty} \int f_n$

$\therefore \limsup_{n\to\infty} \int f_n \leq \int f$

$g + f_n \geq 0$

Fatou $\Rightarrow \int g + f \, d\mu \leq \liminf_{n\to\infty} \int g + f_n = \int g + \liminf_{n\to\infty} \int f_n$

$\therefore \int f \leq \liminf_{n\to\infty} \int f_n \leq \limsup_{n\to\infty} \int f_n \leq \int f$

$\blacksquare$

This is a great theorem.