Math 5210
Tuesday April 21

- Finish Monday discussion for the integration of bounded functions on bounded-measure domains.

We're discussing integral properties. I decided to add a couple more, so rewrote page 2 from Monday:

<u>Theorem 2</u> $(X, \mathcal{F}, \mu)$ a measure space
$\mathcal{F} \ni \mathcal{D}$, $\mu(\mathcal{D}) < \infty$
$g, f : \mathcal{D} \to \mathbb{R}$ bd & measurable. Then

As usual, visualize $(\mathbb{R}, \mathcal{M}, m)$, $\mathcal{M} \ni [a,b]$

(i) $\int_{\mathcal{D}} f + g \, d\mu = \int_{\mathcal{D}} f \, d\mu + \int_{\mathcal{D}} g \, d\mu$ ← did Monday

(ii) $\int_{\mathcal{D}} af \, d\mu = a \int_{\mathcal{D}} f \, d\mu$ ← do $a \geq 0$, $a = -1$, yields general claim

hence (iii) $\int_{\mathcal{D}} af + bg \, d\mu = a \int_{\mathcal{D}} f \, d\mu + b \int_{\mathcal{D}} g \, d\mu$

(iv) $f \leq g$ on $\mathcal{D} \Rightarrow \int_{\mathcal{D}} f \, d\mu \leq \int_{\mathcal{D}} g \, d\mu$ ← use def.

(v) $\left| \int_{\mathcal{D}} f \, d\mu \right| \leq \int_{\mathcal{D}} |f| \, d\mu$ ← use (iv) twice

(vi)(a) Let $A \in \mathcal{F}$, $A \subset \mathcal{D}$. Then $\int_A f \, d\mu = \int_{\mathcal{D}} f \, \mathbb{1}_A \, d\mu$ ← write the def of each and compare!

(vi)(b) $A, B \in \mathcal{F}$, $A, B \subset \mathcal{D}$, $A \cap B = \emptyset$

Then $\int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu$ ← use (vi)(a) and (i)!

(vii) $c_1 \leq f(x) \leq c_2 \Rightarrow c_1 \mu(\mathcal{D}) \leq \int_{\mathcal{D}} f \, d\mu \leq c_2 \mu(\mathcal{D})$

(viii) $A \subset \mathcal{D}$, $\mu(A) = 0 \Rightarrow \int_A f \, d\mu = 0$ ← use def!

(ix) $f = g$ a.e. $\Rightarrow \int_{\mathcal{D}} f \, d\mu = \int_{\mathcal{D}} g \, d\mu$ ← Let $\{x \mid f(x) \neq g(x)\} = \mathcal{B}$, $\mu(\mathcal{B}) = 0$.
Then $\int_{\mathcal{D}} f - g \, d\mu = \int_{\mathcal{D} \setminus \mathcal{B}} f - g \, d\mu + \int_{\mathcal{B}} f - g \, d\mu = 0 + 0.$

(stolen from p. 3 Monday)

Theorem 3 (Bounded convergence theorem) ← Compare to our old uniform convergence theorem, which was the best we could do using Riemann integral ideas.

$(X, \mathcal{F}, \mu)$
$\quad \mathcal{D} \in \mathcal{F}, \ \mu(\mathcal{D}) < \infty$

$\{f_n\}, f : \mathcal{D} \to \mathbb{R}$ uniformly bd & measurable ( $|f_n|, |f| \le M \ \forall x \in \mathcal{D}$ )

If $\{f_n(x)\} \to f(x) \ \forall x \in \mathcal{D}$ (or even just a.e.)

Then
$$\lim_{n\to\infty} \int_{\mathcal{D}} f_n \, d\mu = \int_{\mathcal{D}} f \, d\mu.$$

proof: Let $\varepsilon > 0$. Let $\{f_n(x)\} \to f(x) \ \forall x \in \mathcal{D} \setminus \mathcal{B}$, $\mu(\mathcal{B}) = 0$.

examples?

Consider $E_n := \{ x \in \mathcal{D} \text{ s.t. } |f(x) - f_k(x)| \le \varepsilon \ \forall k \ge n \}$

$E_n \subset E_{n+1}$, $\quad \bigcup E_n \supset \mathcal{D} \setminus \mathcal{B}$, since $\{f_k(x)\} \to f(x) \ \forall x \in \mathcal{D} \setminus \mathcal{B}$.

$\Rightarrow \{\mu(E_n)\} \to \mu(\mathcal{D})$, so $\{\mu(\mathcal{D} \setminus E_n)\} \to 0$

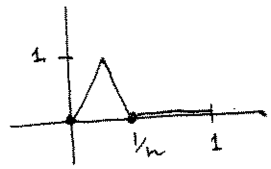
Pick $N$ s.t. $\mu(\mathcal{D} \setminus E_n) \le \varepsilon \ \forall n \ge N$.

Then $n \ge N$

$$\Rightarrow \left| \int_{\mathcal{D}} f \, d\mu - \int_{\mathcal{D}} f_n \, d\mu \right| = \left| \int_{\mathcal{D}} f - f_n \, d\mu \right|$$
$$\le \int_{\mathcal{D}} |f - f_n| \, d\mu$$
$$= \int_{E_n} |f - f_n| \, d\mu + \int_{\mathcal{D} \setminus E_n} |f - f_n| \, d\mu$$
$$\le \varepsilon \mu(E_n) + 2M \mu(\mathcal{D} \setminus E_n)$$
$$\le \varepsilon \left[ \mu(\mathcal{D}) + 2M \right] \qquad \text{"good enough"}$$

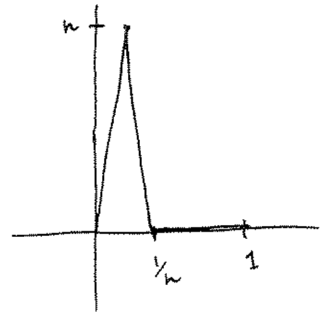
∎

Examples to keep in mind:




$f_n(x) \to 0 \ \forall x \in [0,1]$.
(but not uniformly)
$\{f_n\}$ uniformly bd.

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0.$$

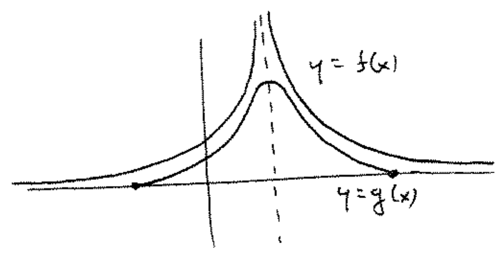



$g_n(x) \to 0 \ \forall x \in [0,1]$.
$\{g_n\}$ not unif bd.
$\frac{1}{2} = \int_0^1 g_n(x)\,dx \not\to \int_0^1 0\,dx$.

## §4.3 Lebesgue integral for non-negative fcns on arbitrary msble domains

**Def**: Let $(X, \widetilde{\mathcal{F}}, \mu)$ meas. space
$f: E \to [0,\infty]$ measurable. ($E \in \widetilde{\mathcal{F}}$)

$$\int_E f\,d\mu := \sup_{\substack{0 \le g \le f \\ g \text{ bounded} \\ g \equiv 0 \text{ outside a} \\ \text{domain } \mathcal{D} \subset E,\ \mu(\mathcal{D}) < \infty}} \int_{\mathcal{D}} g\,d\mu$$




Note: $\int_E f\,d\mu \in [0,\infty]$. Also, by def above, $\int_{\mathcal{D}} g\,d\mu$ also equals $\int_E g\,d\mu$.

properties:

(i) $\int_E cf\,d\mu = c\int_E f\,d\mu$ ($c \ge 0$) ← easy.

(ii) $f = h$ a.e. on $E \Rightarrow \int_E f\,d\mu = \int_E h\,d\mu$ ← by previous Theorem 2(ix), can relax the constraint on "g" in integral def so that $0 \le g \le f$ a.e. then same g's work for f & h.

(iii) $f \le h$ a.e. on $E \Rightarrow \int_E f\,d\mu \le \int_E h\,d\mu$ ← by (ii) can assume $f \le h \ \forall x \in E$.

(iv) $\int_E f + h\,d\mu = \int_E f\,d\mu + \int_E h\,d\mu$ ← this one takes some work.

If $0 \le g_1 \le f$, $0 \le g_2 \le h$ are admissable for $\int_E f\,d\mu$, $\int_E h\,d\mu$, then $0 \le g_1 + g_2 \le f + h$ is admissable for $\int_E f + h\,d\mu$.

Taking the sup's over admissable $g_1, g_2 \Rightarrow$

$$\int_E f\,d\mu + \int_E h\,d\mu \le \int_E (f+h)\,d\mu.$$

Reverse ineq: If

$g_3 \le f+h$, $g_3$ bd, $\equiv 0$ outside set of finite measure

Define $g_1 := \min(g_3, f)$ $\quad \left( = \frac{g_3+f}{2} - \frac{|g_3-f|}{2} \text{ is msble} \right)$

$g_2 = g_3 - g_1$

Check: $g_1$ is admissable for $f$:

$0 \le g_1 \le f$

$0 \le g_1 \le g_3$.

Also, $g_2$ is admissable for $h$!

$0 \le g_3 - g_1 \le g_3$

and,

if $g_3(x) \le f(x)$

then $g_2 = g_3 - g_3 = 0 \le h$

if $g_3(x) > f(x)$, then

$g_2(x) = g_3(x) - f(x)$

$\le f + h - f = h(x)$

$g_1 + g_2 = g_3$

$$\int_E g_3\,d\mu = \int_E g_1\,d\mu + \int_E g_2\,d\mu$$

$$\le \int_E f\,d\mu + \int_E h\,d\mu$$

take sup of LHS over admissable $g_3 \Rightarrow$ $\int_E f+h\,d\mu \le \int_E f\,d\mu + \int_E h\,d\mu$ ! ∎

(v) $A \subset E \Rightarrow \int_A f\,d\mu = \int_E f \mathbb{1}_A\,d\mu$ : use def; every admissable $g$ on the right corresponds to one on the left & vise versa.

(vi) $A, B \subset E$, $A \cap B = \emptyset$ $\Rightarrow \int_{A \cup B} f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu$ : use (v)

tomorrow: Fatou's Lemma & the monotone convergence theorem.

4.4 General Lebesgue integrable fcns, and the dominated convergence theorem