Math 5210
Monday April 20
Royden 4.2 finished?

We're discussing the Lebesgue integral of bounded fcns on finite measure domains:

$$(X, \mathcal{F}, \mu)$$
$$\cup$$
$$\mathcal{D},\ \mu(\mathcal{D}) < \infty$$
$$f: \mathcal{D} \to \mathbb{R},\quad |f| \le M.$$

On Friday we showed that f <u>measurable</u> $\Rightarrow \int_{\mathcal{D}} f\,d\mu$ exists, i.e. is the common value of
$$\sup_{\substack{\varphi \le f \\ \varphi \text{ simple}}} \int_{\mathcal{D}} \varphi\,d\mu = \inf_{\substack{\chi \ge f \\ \chi \text{ simple}}} \int_{\mathcal{D}} \psi\,d\mu.$$

$$\boxed{\text{• where did the hypothesis that } f \text{ is measurable get used?}}$$

• Show reverse statement, p. 4 Friday:

If $\quad \underline{\mathcal{L}} := \sup_{\substack{\varphi \le f \\ \varphi \text{ simp}}} \int_{\mathcal{D}} \varphi\,d\mu = \inf_{\substack{f \le \chi \\ \chi \text{ simp}}} \int_{\mathcal{D}} \chi\,d\mu := \overline{\mathcal{L}}$, i.e. $\int_{\mathcal{D}} f\,d\mu$ exists.

Then f is measurable!

then,

<u>Theorem 1</u>: Let $f: [a,b] \to \mathbb{R}$ be Riemann integrable (hence bounded), i.e.

$$\underbrace{\sup_{\substack{\varphi \le f \\ \varphi \text{ step fcn}}} \int_a^b \varphi\,dx}_{\substack{\text{this is the sup of the lower sums for } f\text{, i.e.} \\ \sup_P \left\{\sum_{i=1}^n m_i \Delta x_i\right\} \\ m_i = \inf_{x \in I_i} \{f(x)\}}} = \underbrace{\inf_{\substack{f \le \chi \\ \chi \text{ step fcn}}} \int_a^b \chi\,dx}_{\text{inf of upper sums.}} := R\int_a^b f(x)\,dx$$

temporary notation for Riemann integral

Then The Lebesgue integral $\int_a^b f(x)\,dx$ exists $\qquad (\mathbb{R}, \mathcal{M}, m)$
and equals the Riemann integral.

<u>Proof</u>: sandwich:
$$\sup_{\substack{\varphi \le f \\ \varphi \text{ step}}} \int_a^b \varphi\,dx \le \sup_{\substack{\varphi \le f \\ \varphi \text{ simple}}} \int_a^b \varphi\,dx \le \inf_{\substack{f \le \chi \\ \chi \text{ simple}}} \int_a^b \chi\,dx \le \inf_{\substack{f \le \psi \\ \psi \text{ step}}} \int_a^b \psi\,dx \quad \blacksquare$$

Theorem 2 $(X, \mathcal{F}, \mu)$ measure space

$\mathcal{D} \in \mathcal{F}$, $\mu(\mathcal{D})<\infty$

$g, f : \mathcal{D} \to \mathbb{R}$ bounded & measurable. Then

(i) $\displaystyle\int_{\mathcal{D}} f+g \, d\mu = \int_{\mathcal{D}} f \, d\mu + \int_{\mathcal{D}} g \, d\mu$

(ii) $\displaystyle\int_{\mathcal{D}} af \, d\mu = a \int_{\mathcal{D}} f \, d\mu$

(i), (ii) $\Rightarrow$ (iii) $\displaystyle\int_{\mathcal{D}} af+bg \, d\mu = a\int_{\mathcal{D}} f \, d\mu + b\int_{\mathcal{D}} g \, d\mu$

(iv) $f \le g$ on $\mathcal{D}$ $\Rightarrow \displaystyle\int_{\mathcal{D}} f \, d\mu \le \int_{\mathcal{D}} g \, d\mu$

(v) $\displaystyle\left|\int_{\mathcal{D}} f \, d\mu\right| \le \int_{\mathcal{D}} |f| \, d\mu$

(vi) $A, B \in \mathcal{F}$, $A \cup B \subset \mathcal{D}$, $A \cap B = \emptyset$

$\displaystyle\Rightarrow \int_{A\cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu$

(vii) $c_1 \le f(x) \le c_2$ a.e. on $\mathcal{D}$

$\displaystyle\Rightarrow c_1 \mu(\mathcal{D}) \le \int_{\mathcal{D}} f \, d\mu \le c_2 \mu(\mathcal{D})$.

(i): $\displaystyle\int_{\mathcal{D}} f+g \, d\mu = \sup_{\substack{\varphi \le f+g \\ \varphi \text{ simp}}} \int_{\mathcal{D}} \varphi \, d\mu$

$\displaystyle\ge \sup_{\substack{\varphi_1 \le f \\ \varphi_2 \le g}} \int_{\mathcal{D}} \varphi_1 + \varphi_2 \, d\mu$

$\displaystyle= \sup_{\varphi_1 \le f} \int_{\mathcal{D}} \varphi_1 \, d\mu + \sup_{\varphi_2 \le g} \int_{\mathcal{D}} \varphi_2 \, d\mu$

$\displaystyle= \int_{\mathcal{D}} f \, d\mu + \int_{\mathcal{D}} g \, d\mu$

get reverse ineq. using inf characterization.
$f+g \le \chi$

(ii) $a=0$ clear.

$a>0$: $\displaystyle\int_{\mathcal{D}} af \, d\mu = \sup_{\substack{\varphi \le af \\ \varphi \text{ simp}}} \int_{\mathcal{D}} \varphi \, d\mu$

$\displaystyle= \sup_{\substack{\frac{\varphi}{a} \le f \\ \varphi \text{ simp}}} a \int_{\mathcal{D}} \frac{\varphi}{a} \, d\mu$

$\displaystyle= \sup_{\substack{\tilde\varphi \le f \\ \tilde\varphi \text{ simp}}} a \int_{\mathcal{D}} \tilde\varphi \, d\mu$

$\displaystyle= a \int_{\mathcal{D}} f \, d\mu$

$a=-1$:

$\displaystyle\int_{\mathcal{D}} -f \, d\mu = \sup_{\substack{\varphi \le -f \\ \varphi \text{ simp}}} \int_{\mathcal{D}} \varphi \, d\mu$

$\displaystyle= \sup_{\substack{-\varphi \ge f \\ \varphi \text{ simp}}} - \int_{\mathcal{D}} -\varphi \, d\mu$

$\displaystyle= -\inf_{\substack{\chi \ge f \\ \chi \text{ simp}}} \int_{\mathcal{D}} \chi \, d\mu$

$\displaystyle= -\int_{\mathcal{D}} f \, d\mu$

$\Rightarrow$ general a

(iv) $f \le g$ on $\mathcal{D}$ $\Rightarrow \displaystyle\int_{\mathcal{D}} (f-g) \, d\mu = \inf_{\substack{f-g \le \chi \\ \chi \text{ simp}}} \int_{\mathcal{D}} \chi \, d\mu \le 0$ (take $\chi \equiv 0$)

$\parallel$

$\displaystyle\int_{\mathcal{D}} f \, d\mu - \int_{\mathcal{D}} g \, d\mu$ by (iii) ■

(v): $f \le |f| \Rightarrow \displaystyle\int_{\mathcal{D}} f \, d\mu \le \int_{\mathcal{D}} |f| \, d\mu$. $\quad -f \le |f| \Rightarrow -\displaystyle\int_{\mathcal{D}} f \, d\mu \le \int_{\mathcal{D}} |f| \, d\mu$.

■

(vi) $$\int_{A\cup B} f\,d\mu = \int_X f\mathbb{1}_{A\cup B}\,d\mu = \int_X f(\mathbb{1}_A + \mathbb{1}_B)\,d\mu \quad \text{since } A\cap B = \emptyset$$

$$= \int_X f\mathbb{1}_A\,d\mu + \int_X f\mathbb{1}_B\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu.$$

(vii) $$c_1\mu(\mathcal{D}) = \int_{\mathcal{D}} c_1\,d\mu \le \sup_{\substack{\varphi\le f \text{ (on } \mathcal{D})\\ \varphi \text{ simple}}} \int_{\mathcal{D}} \varphi\,d\mu = \int_{\mathcal{D}} f\,d\mu = \inf_{\substack{f\le \chi\\ \chi \text{ simple}}} \int_{\mathcal{D}} \chi\,d\mu \le \int_{\mathcal{D}} c_2\,d\mu = c_2\mu(\mathcal{D})$$

$\blacksquare$

<u>Theorem 3</u> (<u>Bounded convergence theorem</u>) ← Compare to our old uniform convergence theorem, which was the best we could do using Riemann integral ideas.

$(X, \mathcal{F}, \mu)$

$\mathcal{F} \ni \mathcal{D}$, $\mu(\mathcal{D}) < \infty$

$\{f_n\}, f : \mathcal{D} \to \mathbb{R}$ uniformly bd & measurable ($|f_n|, |f| \le M \;\forall x\in\mathcal{D}$)

If $\{f_n(x)\} \to f(x) \;\forall x \in \mathcal{D}$ (or even just a.e.)

Then
$$\lim_{n\to\infty} \int_{\mathcal{D}} f_n\,d\mu = \int_{\mathcal{D}} f\,d\mu.$$

<u>proof</u>: Let $\varepsilon > 0$. Let $\{f_n(x)\} \to f(x) \;\forall x \in \mathcal{D}\setminus\mathcal{B}$, $\mu(\mathcal{B}) = 0$.

examples?

Consider $E_n := \{x \in \mathcal{D} \text{ s.t. } |f(x) - f_k(x)| \le \varepsilon \;\forall k \ge n\}$

$E_n \subset E_{n+1}$, $\bigcup E_n \supset \mathcal{D}\setminus\mathcal{B}$, since $\{f_k(x)\} \to f(x) \;\forall x\in \mathcal{D}\setminus\mathcal{B}$.

$\Rightarrow \{\mu(E_n)\} \to \mu(\mathcal{D})$, so $\{\mu(\mathcal{D}\setminus E_n)\} \to 0$

Pick N s.t. $\mu(\mathcal{D}\setminus E_n) \le \varepsilon \;\forall n \ge N$.

Then $n \ge N$

$$\Rightarrow \left|\int_{\mathcal{D}} f\,d\mu - \int_{\mathcal{D}} f_n\,d\mu\right| = \left|\int_{\mathcal{D}} f - f_n\,d\mu\right|$$

$$\le \int_{\mathcal{D}} |f - f_n|\,d\mu$$

$$= \int_{E_n} |f-f_n|\,d\mu + \int_{\mathcal{D}\setminus E_n} |f - f_n|\,d\mu$$

$$\le \varepsilon\mu(E_n) + 2M\mu(\mathcal{D}\setminus E_n)$$

$$\le \varepsilon[\mu(\mathcal{D}) + 2M]$$ "good enough"

$\blacksquare$