Math 5210
Friday April 17

Final HW assignment is due Monday April 27
Royden p. 86 # 3, 4, 7, 8
p 89–90 # 10, 15, 16, 17, 18, 19.
(p 70 # 28 is canceled :-))

(1)

- We were in the middle of proving

Theorem 1: Let φ, χ be integrable simple functions, on the measure space $(X, \mathcal{F}, \mu)$
(think $(\mathbb{R}, \mathcal{M}, m)$)

A) Then so is the linear combination

$c\varphi + d\chi$.

In fact, $$\int_X c\varphi + d\chi \, d\mu = c\int_X \varphi \, d\mu + d\int_X \chi \, d\mu.$$

B) If $\varphi \le \chi$ then $\int \varphi \, d\mu \le \int \chi \, d\mu$.

B) follows from A), since $\chi - \varphi \geq 0$ is a simple function, so $\int \chi - \varphi \, d\mu \geq 0$ by def'n.
Therefore $\int \chi - \int \varphi \geq 0$ by (A) ■

proof of A):

Write $\varphi = \sum_{j=1}^{n} a_j \mathbb{1}_{A_j}$, $\chi = \sum_{i=1}^{m} b_i \mathbb{1}_{B_i}$ in canonical form.

($A_j = \varphi^{-1}(\{a_j\})$, $\{a_j\}$ distinct and $\neq 0$, $\mu(A_j) < \infty$ ... analogous for χ.)

Recall, we also have defined
$\varphi^{-1}(\{0\}) = A_0 = X \setminus (\bigcup_{j=1}^{n} A_j)$, $a_0 = 0$
$\chi^{-1}(\{0\}) = B_0 = X \setminus (\bigcup_{i=1}^{m} B_i)$, $b_0 = 0$

So $\{A_j\}_{j=0}^{n}$, $\{B_i\}_{i=0}^{m}$ partition X.

Therefore $\{A_j \cap B_i\}_{\substack{j=1..n \\ i=1..m}}$ is a common subpartition of $\{A_j\}$ and $\{B_i\}$.

$$\mathbb{1}_{A_j} = \sum_{i=0}^{m} \mathbb{1}_{A_j \cap B_i} ; \quad \mathbb{1}_{B_i} = \sum_{j=0}^{n} \mathbb{1}_{B_i \cap A_j}.$$

So,
$$\varphi = \sum_{j=1}^{n} \sum_{i=0}^{m} a_j \mathbb{1}_{A_j \cap B_i}, \quad \chi = \sum_{i=1}^{m} \sum_{j=0}^{n} b_i \mathbb{1}_{B_i \cap A_j}$$

or $j=0$, since $a_0 = 0$ — or $i=0$, since $b_0 = 0$

Therefore

$$c\varphi + d\chi = \sum_{\substack{j=0..n \\ i=0..m \\ (i,j)\neq(0,0)}} (ca_j + db_i)\, \mathbb{1}_{A_j \cap B_i}$$

$(i,j)\neq(0,0) \longleftarrow (i,j)=(0,0)$ contributes nothing to the sum since $a_0 = b_0 = 0$.
Omitting this term guarantees that all the other $\mu(A_j \cap B_i) < \infty$.

By the lemma we proved Wednesday,

$$\int c\varphi + d\chi \, d\mu = \sum_{\substack{j=0..n \\ i=0..m \\ (i,j)\neq(0,0)}} (ca_j + db_i)\, \mu(A_j \cap B_i)$$

$$= c \sum_{j=1}^{n} a_j \underbrace{\sum_{i=0}^{m} \mu(A_j \cap B_i)}_{\mu(A_j)\text{, since } \{B_i\}_{i=0}^{m} \text{ is a measurable partition of } X.} + d \sum_{i=1}^{n} b_i \underbrace{\sum_{j=0}^{n} \mu(A_j \cap B_i)}_{\mu(B_i)}$$

$$= c\int \varphi \, d\mu + d \int \chi \, d\mu \quad \blacksquare$$

(Corollary: if $\varphi = \sum_{i=1}^{n} c_i \mathbb{1}_{E_i}$, with $\mu(E_i) < \infty$, then $\int \varphi \, d\mu = \sum_{i=1}^{n} c_i \mu(E_i)$, whether the $\{E_i\}$ are disjoint or not.)

Definition: Let $(X, \mathcal{F}, \mu)$ be a measure space, $\mathcal{D} \in \mathcal{F}$, $\mu(\mathcal{D}) < \infty$

Think $(\mathbb{R}, \mathcal{M}, m)$, $[a,b] \in \mathcal{M}$. $f : [a,b] \to \mathbb{R}$, $|f| \le M$.

Let $f : \mathcal{D} \to \mathbb{R}$ be bounded.
By Theorem 1B),

$$\sup_{\substack{\varphi \le f \\ \varphi \text{ simple}}} \int_{\mathcal{D}} \varphi \, d\mu \le \inf_{\substack{\chi \ge f \\ \chi \text{ simple}}} \int_{\mathcal{D}} \chi \, d\mu$$

We say that $\int_{\mathcal{D}} f \, d\mu$ exists (or f is integrable on $\mathcal{D}$) iff the above inequality is an equality, and in this case define $\int_{\mathcal{D}} f \, d\mu$ to be this common value.

$$\boxed{\begin{array}{l}\text{Theorem 2: Let } (X, \mathcal{F}, \mu), \mathcal{D}, f \text{ be as above.} \\ \text{Then } f \text{ is \underline{measurable} iff } f \text{ is \underline{integrable}}\end{array}}$$

(actually, integrable $\Rightarrow$ measurable assumes $\mathcal{F}$ is the σ-algebra of measurable sets arising from an outer measure) proof: $\Rightarrow$: let f be measurable; $|f| \leq M$.

Pick large $n \in \mathbb{N}$ ($\frac{1}{n}$ is range partition size).

For $j \in \mathbb{N}$, $-M \leq \frac{j}{n} \leq M$

let $I_j = [\frac{j}{n}, \frac{j+1}{n})$.

let $A_j = f^{-1}(I_j) \in \widetilde{\mathcal{F}}$, since we're assuming f is measurable.

$= \{x \in \mathcal{D} \text{ s.t. } j/n \leq f(x) < \frac{j+1}{n}\}$. $\{A_j\}$ partition $\mathcal{D}$.

Then

$$\varphi_n := \sum_j j/n \, \mathbb{1}_{A_j} \leq f \leq \sum_j \frac{j+1}{n} \mathbb{1}_{A_j} := \chi_n$$

So

$$\int_{\mathcal{D}} \varphi_n \, d\mu \leq \underbrace{\sup_{\substack{\varphi \leq f \\ \varphi \text{ simple}}} \int_{\mathcal{D}} \varphi \, d\mu}_{:= \underline{I}} \leq \underbrace{\inf_{\substack{\chi \geq f \\ \chi \text{ simple}}} \int_{\mathcal{D}} \chi \, d\mu}_{:= \overline{I}} \leq \int_{\mathcal{D}} \chi_n \, d\mu$$

So $\quad \overline{I} - \underline{I} \leq \int_{\mathcal{D}} (\chi_n - \varphi_n) \, d\mu = \sum_j \frac{1}{n} \mu(A_j) = \frac{1}{n} \mu(\mathcal{D}) \quad$ since $\{A_j\}$ partition $\mathcal{D}$.

let $n \to \infty$, $\Rightarrow \overline{I} - \underline{I} \leq 0$ ∎

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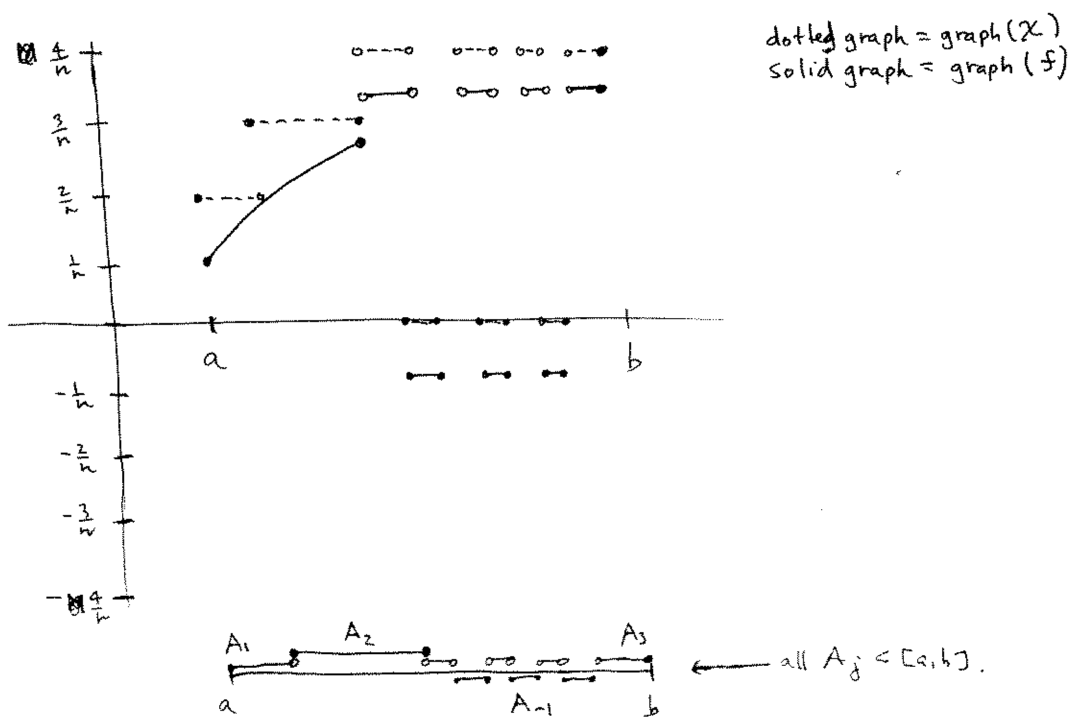
optional illustration



$\Leftarrow$: Let $\displaystyle\int_{\mathcal{D}} f\,d\mu = \sup_{\substack{\varphi \le f \\ \varphi \text{ simple}}} \int_{\mathcal{D}} \varphi\,d\mu = \inf_{\substack{f \le \chi \\ \chi \text{ simple}}} \int_{\mathcal{D}} \chi\,d\mu := \mathcal{L}$. Show f is measurable

Pick φ_n as above $\left\{ \int_{\mathcal{D}} \varphi_n\,d\mu \right\} \to \mathcal{L}$

χ_n as above, $\left\{ \int_{\mathcal{D}} \chi_n\,d\mu \right\} \to \mathcal{L}$

Let $\varphi_* := \sup_n \varphi_n \le f$

$\chi_* = \inf_n \chi_n \geq f$

<u>We claim $\varphi_* = f = \chi_*$</u> a.e., which will prove f is measurable.

$$\{x \in \mathcal{D} \text{ s.t. } \chi_*(x) > \varphi_*(x)\} = \bigcup_{m \in \mathbb{N}} \underbrace{\{x \mid \chi_*(x) > \varphi_*(x) + \tfrac{1}{m}\}}_{:= E_m}$$

suffices to show $\mu(E_m) = 0$ $\forall m$, since then $\chi_* = \varphi_*$ a.e.

But $\chi_*(x) > \varphi_*(x) + \frac{1}{m}$

$\Rightarrow \chi_n(x) > \varphi_n(x) + \frac{1}{m}$ (with $\chi_n \geq \chi_*$, $\varphi_n \leq \varphi_*$)

So $E_m \subset F_m^{(n)} = \{x \mid \chi_n(x) > \varphi_n(x) + \frac{1}{m}\}$

Notice,

$$\int_{\mathcal{D}} \chi_n - \varphi_n \overset{(1B)}{\geq} \int_{F_m^{(n)}} \chi_n - \varphi_n\,d\mu \overset{(1B)}{\geq} \frac{1}{m}\mu(F_m^{(n)})$$

$$\| $$

$$\int_{\mathcal{D}} \chi_n - \int_{\mathcal{D}} \varphi_n\,d\mu$$

$$\downarrow n \to \infty$$

$$\mathcal{L} - \mathcal{L} = 0$$

deduce $\lim_{n \to \infty} \mu(F_m^{(n)}) = 0$

But $\mu(E_m) \le \mu(F_m^{(n)})$ $\forall n$

So $\mu(E_m) = 0$

On Monday we continue — read Royden, we're about to prove that Riemann integrable fns on $[a,b]$ are Lebesgue integrable (measure m), with the integral values the same.

This is page 79. We should finish 4.2 on Monday,
4.3 Tuesday
4.4 Wednesday.