Math 5210
Wednesday April 15

①

Chapter 4 Lebesgue Integral.

4.1

first, recall Riemann integral for $f: [a,b] \to \mathbb{R}$.

Our Def (So that we could consider $f: [a,b] \to X_{\text{Banach}}$)

$$\int_a^b f(x) = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

P: $a = x_0 < x_1 < \dots < x_n = b$

$x_i^* \in I_i = [x_{i-1}, x_i]$

$\Delta x_i = x_i - x_{i-1}$

(integral $\exists$ iff lim $\exists$)

Royden Def, for $f: [a,b] \to \mathbb{R}$.

For a partition as above,

let $M_i = \sup_{x \in I_i} \{f(x)\}$

$m_i = \inf_{x \in I_i} \{f(x)\}$

$S = R_P^+ := \sum_{i=1}^{n} M_i \Delta x_i$ upper sum

$s := R_P^- = \sum_{i=1}^{n} m_i \Delta x_i$ lower sum

By a common refinement argument,
every upper sum (for any partition) is greater than every lower sum (for any other partition)

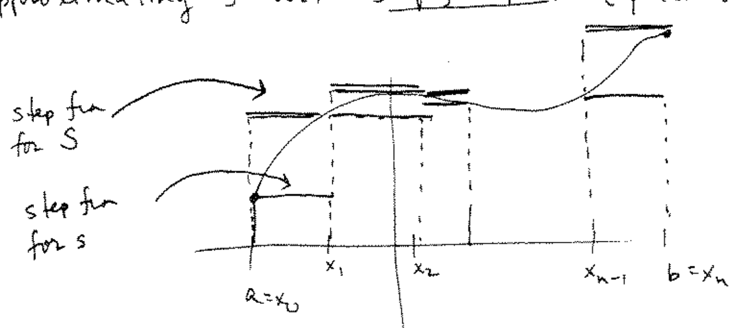
so $\inf_P \{S\} \geq \sup_P \{s\}$

We'll compare this to the Lebesgue integral.

$$\boxed{\int_a^b f(x)\,dx \ \exists \text{ iff } \inf_P \{S\} = \sup_P \{s\}, \text{ and its value is this common number.}}$$

It is a "fun" exercise (which you don't have to hand in), to show that "Our def" and "Royden def" of Riemann integrable (and integral) are in fact equivalent... it's not immediately obvious!

In either case, your integral is defined by partioning the domain, and approximating f with step functions (piecewise const. on subintervals).



← partitioned the domain into subintervals

§4.2 : Lebesgue integral for bounded measurable fcns, on domains of finite measure.

Def Let $(X, \tilde{\mathcal{F}}, \mu)$ a measure space

For $E \in \tilde{\mathcal{F}}$, the characteristic fcn

$$\chi_E(x) = \mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$$

A simple fcn $\varphi(x)$ is a finite linear combo of characteristic fcns

$$\varphi(x) = \sum_{i=1}^{m} b_i \mathbb{1}_{E_i}(x) \qquad E_i \in \tilde{\mathcal{F}} \quad 1 \le i \le m.$$

Notice, a measurable $\varphi : X \to \mathbb{R}$ is a simple function iff its range consists of only finitely many values $\{a_0, a_1, a_2 .. a_n\} \subset \mathbb{R}$, and we can always express φ as

$$\varphi(x) = \sum_{j=1}^{n} a_j \mathbb{1}_{A_j}(x) \qquad A_j = \varphi^{-1}(\{a_j\})$$ ← canonical representation

example : $2 \mathbb{1}_E + 3 \mathbb{1}_F = 5 (\mathbb{1}_{E \cup F}) + 2 (\mathbb{1}_{E \setminus F}) + 3 \mathbb{1}_{F \setminus E}$

Def If φ is a simple fcn, zero outside of a finite measure set and with canonical representation

$$\varphi(x) = \sum_{j=1}^{n} a_j \mathbb{1}_{A_j}(x)$$

then
$$\int_X \varphi \, d\mu := \sum_{j=1}^{n} a_j \mu(A_j)$$

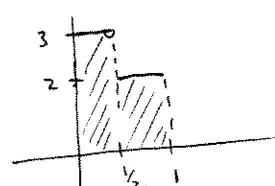
If $\mathcal{D} \in \tilde{\mathcal{F}}$, then
$$\int_{\mathcal{D}} \varphi \, d\mu := \int_X \varphi \mathbb{1}_{\mathcal{D}} \, d\mu = \sum_{j=1}^{n} a_j \mu(A_j \cap \mathcal{D}).$$

example $(\mathbb{R}, \mathcal{M}, m)$

$$\varphi(x) = 3 \mathbb{1}_{[0, 1/2)} + 2 \mathbb{1}_{[1/2, 1]}$$

$$\int_{\mathbb{R}} \varphi \, dx = 3 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{2} = 5/2$$

↑ rather than dm



(step functions are simple fcns, and this formula reproduces the Riemann sum.... simple fcns can be much wilder than step fcns, however.)

Algebra for simple fcns and their integrals

Lemma 1 Let $\varphi(x) = \sum_{j=1}^{n} a_j \mathbb{1}_{A_j}(x)$ $\qquad A_j = \varphi^{-1}(\{a_j\}) \qquad (a_j \neq 0)$

in canonical form, with $\mu(A_j) < \infty \;\forall j$.

If also
$$\varphi(x) = \sum_{i=1}^{m} b_i \mathbb{1}_{E_i}(x) \qquad \text{with } \{E_i\} \subset \mathcal{F}, \text{ a disjoint collection of finite-measure sets.}$$
(so the $\{E_i\}$ are refining the $\{A_j\}$)

Then $$\int_X \varphi \, d\mu = \sum_{j=1}^{n} a_j \mu(A_j) = \sum_{i=1}^{m} b_i \mu(E_i)$$

proof: Since $A_j = \varphi^{-1}(\{a_j\})$, and using the 2^{nd} formula for φ,
$$A_j = \bigcup_{\substack{i \text{ s.t.} \\ b_i = a_j}} E_i \quad \Rightarrow \mu(A_j) = \sum_{\substack{i \text{ s.t.} \\ b_i = a_j}} \mu(E_i)$$
$$\Rightarrow \quad a_j \mu(A_j) = \sum_{\substack{i \text{ s.t.} \\ b_i = a_j}} b_i \mu(E_i)$$
$$\Rightarrow \quad \sum_j a_j \mu(A_j) = \sum_j \Big(\sum_{\substack{i \text{ s.t.} \\ b_i = a_j}} b_i \mu(E_i) \Big) = \sum_i b_i \mu(E_i). \qquad \blacksquare$$

Theorem 1 Let φ, χ be integrable simple functions.

A) Then so is
$$c\varphi + d\chi.$$
In fact, $$\int_X c\varphi + d\chi \, d\mu = c \int_X \varphi \, d\mu + d \int_X \chi \, d\mu$$ (Integration of simple fcns is linear)

B) If $\varphi \geq 0$, then $\int_X \varphi \, d\mu \geq 0$.

B) is immediate, because if φ is written in canonical form all the $a_i > 0$. $\blacksquare$

A). Write $\varphi = \sum_{j=1}^{n} a_j \mathbb{1}_{A_j}$, $\chi = \sum_{i=1}^{m} b_i \mathbb{1}_{B_i}$ in canonical form.

Define $A_0 = X \setminus (\bigcup_{j=1}^{n} A_j)$; $B_0 = X \setminus (\bigcup_{i=1}^{m} B_i)$, so $\{A_j\}_{j=0}^{n}$, $\{B_i\}_{i=0}^{m}$ partition X.

Therefore, $\mathbb{1}_{A_j} = \sum_{i=0}^{m} \mathbb{1}_{A_j \cap B_i}$; $\mathbb{1}_{B_i} = \sum_{j=0}^{n} \mathbb{1}_{A_j \cap B_i}$. (So also $\{A_j \cap B_i\}_{\substack{j=0..n \\ i=0..m}}$)

Define $a_0 = b_0 = 0$, as before.

Thus $$c\varphi + d\chi = \sum_{\substack{j=0..n \\ i=0..m}} (ca_j + db_i) \mathbb{1}_{A_j \cap B_i}.$$

Thus by lemma 1,

$$\int_X c\varphi + d\chi \, d\mu = \sum_{(i,j)\neq(0,0)} (ca_j + db_i)\, \mu(A_j \cap B_i)$$

$$= c\sum_{j\neq 0} a_j \mu(A_j) + d\sum_{i\neq 0} b_i \mu(B_i)$$

since e.g. $\mu(A_j) = \sum_{i=0}^{m} \mu(A_j \cap E_i)$.

$$= c\int \varphi \, d\mu + d\int \chi \, d\mu \qquad \blacksquare$$

<u>Theorem 2</u> (Analogous to page 1 result for the Riemann integral, but at the same time <u>this</u> theorem completely characterizes the (bounded) Lebesgue Integrable fns.)

Let $(X, \mathcal{F}, \mu)$ a measure space
$\mathcal{D} \in \mathcal{F}$, $\mu(\mathcal{D}) < \infty$.

Think of $(\mathbb{R}, \mathcal{M}, m)$, $[a,b] \in \mathcal{M}$.

Let $f : \mathcal{D} \to \mathbb{R}$ be bounded.

Then by Theorem 1b),

$$\sup_{\substack{\varphi \le f \text{ (on } \mathcal{D}) \\ \varphi \text{ simple}}} \int_{\mathcal{D}} \varphi(x)\, d\mu \le \inf_{\substack{\chi \ge f \\ \chi \text{ simple}}} \int_{\mathcal{D}} \chi \, d\mu$$

If this inequality is an equality, we say that $\int_{\mathcal{D}} f \, d\mu$ exists, and assign its value to be this common value.

f is Lebesgue integrable on $\mathcal{D}$.

Furthermore, (bounded) <u>f is integrable</u>
<u>if and only if f is measurable</u>

proof next time!
We shall partition the range!