postpone p. 86 HW to next week. 1

Math 5210

Tuesday April 14. ~ Finish chapter 3 today, begin chapter 4 tomorrow.

- Finish Monday notes starting page 3.5: Convergence & measurable functions.

- §3.6: Littlewood's 3 Principles: (for $(X, \mathcal{F}, \mu) = (\mathbb{R}, \mathcal{M}, m)$)

  1) Every measurable set is nearly a finite union of intervals

     (You proved in HW that E measurable, $m(E) < \infty$ iff $\forall \varepsilon > 0 \ \exists \ \mathcal{U} = I_1 \cup I_2 \cup \dots \cup I_n$ s.t. $m((E \setminus \mathcal{U}) \cup (\mathcal{U} \setminus E)) < \varepsilon$).

  2) Every measurable function is nearly continuous

     (Thm later: Let $f: \mathbb{R} \to \mathbb{R}$ measurable. Then $\forall \varepsilon > 0 \ \exists \ E \in \mathcal{M}$ with $m(E) < \varepsilon$, s.t. $f\big|_{\mathbb{R} \setminus E}$ is continuous.)

  3) Every convergent (ptwise) sequence of measurable functions is nearly uniformly convergent:

<u>Theorem (Egoroff)</u>: Let $(X, \mathcal{F}, \mu)$ be a measure space.

Let $\mathcal{D} \in \mathcal{F}$, $\{f_k\}, f: \mathcal{D} \to \mathbb{R}$ (not $\overline{\mathbb{R}}$), $\underline{\mu(\mathcal{D}) < \infty}$.

Let $\lim_{k \to \infty} f_k(x) = f(x)$ a.e. (i.e. $\forall x \in \mathcal{D} \setminus \mathcal{B}$, $\mu(\mathcal{B}) = 0$).

Then $\forall \delta > 0 \ \exists \ E_\delta \subset \mathcal{D}$, $\mu(\mathcal{D} \setminus E_\delta) < \delta$ s.t. $\{f_k\} \to f$ <u>uniformly</u> on E_δ!

<u>proof</u>: Replace $\mathcal{D}$ with $\mathcal{D} \setminus \mathcal{B}$ (and call it $\mathcal{D}$), so that $\{f_k(x)\} \to f(x) \ \forall x \in \mathcal{D}$.

<u>Step 1</u> Let $\varepsilon > 0$. Define $F_m^\varepsilon = \{x \in \mathcal{D} \text{ s.t. } |f_k(x) - f(x)| < \varepsilon \ \forall k \geq m\}$

Then $F_m^\varepsilon \subset F_{m+1}^\varepsilon$

and $\bigcup_m F_m^\varepsilon = \mathcal{D}$, since $\{f_k(x)\} \to f(x) \ \forall x \in \mathcal{D}$!

$\Rightarrow \{\mu(F_m^\varepsilon)\} \to \mu(\mathcal{D}) = \mu(F_m^\varepsilon) + \mu(\mathcal{D} \setminus F_m^\varepsilon)$

and so,

$\{\mu(\mathcal{D} \setminus F_m^\varepsilon)\} \to 0.$

$\therefore \ \forall \rho > 0 \ \exists \ M = M(\rho)$ s.t. $\mu(\mathcal{D} \setminus F_M^\varepsilon) < \rho$.

Step 2 Let $\delta > 0$.

$\forall n \in \mathbb{N}$ use Step 1, to find a set

$$F^{1/n}_{M_n} = \{x \in \mathcal{D} \text{ s.t. } |f_k(x) - f(x)| < \tfrac{1}{n} \ \forall k \geq M_n\}$$

$$\mu(\mathcal{D} \setminus F^{1/n}_{M_n}) < \delta/2^n.$$

Let $E_\delta := \bigcap_n F^{1/n}_{M_n}$

- $\forall x \in E_\delta$, $n \in \mathbb{N}$, $\exists M_n$ s.t. $k \geq M_n \Rightarrow |f_k(x) - f(x)| < 1/n$

  so $\{f_k\} \to f$ uniformly on E_δ

in $\mathcal{D}$,

- $E_\delta^c = \bigcup_n (F^{1/n}_{M_n})^c \Rightarrow \mu(E_\delta^c) \leq \sum_{n=1}^{\infty} \delta/2^n = \delta$ ∎

Compositions of measurable fns may not be measurable! (See HW.)

$(\mathbb{R}, \mathcal{M}, m)$

You may appeal to §3.4, $\exists$ a non-measurable set for Lebesgue outer measure m^*

Proof: Let $E \subset [0,1]$. For $y \in [0,1)$ define

$$E \dotplus y = \{x \in [0,1] \text{ s.t. } x = e + y \text{ for } e \in E, \text{ or } x = e + y - 1 \text{ for } e \in E\}$$

(you are translating E by y, mod 1).

Lemma: $E \in \mathcal{M} \Rightarrow E \dotplus y \in \mathcal{M}$ and $m(E \dotplus y) = m(E)$

proof Let $0 \leq y < 1$. $E_1 := E \cap [0, 1-y)$
$E_2 : E \cap [1-y, 1]$

$$\left.\begin{array}{l} E_1 \dotplus y = E_1 + y \in \mathcal{M} \\ E_2 \dotplus y = E_2 + y - 1 \in \mathcal{M} \end{array}\right\} \text{ so } E \dotplus y = (E_1 + y) \cup (E_2 + y - 1) \in \mathcal{M} \text{ and } m(E \dotplus y) = m(E_1) + m(E_2) = m(E).$$

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Now consider the equivalence relation on $[0,1)$, $x \sim y$ iff $x - y \in \mathbb{Q}$ the rational #'s.
This is an equivalence relation, so partitions $[0,1)$ into equivalence classes,
$[0,1) = \bigcup_{a \in A} E_a$. By the axiom of choice $\exists$ a set P s.t. P contains exactly one element from each equivalence class. This implies

$$[0,1) = \bigcup_{q \in \mathbb{Q} \cap [0,1)} P \dotplus q \qquad \text{disjoint union!!}$$

If P were measurable, we'd have $1 = m([0,1)) = \sum_{q \in \mathbb{Q} \cap [0,1)} m(P)$ fails! $\cancel{m(P) = 0}$ $\cancel{m(P) > 0}$.

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