Math 5210
Monday April 13

<u>Measurable functions</u> §3.5 Royden.

Let μ be a <u>measure</u>, on a <u>σ-algebra</u> $\mathcal{F}$ of subsets of a space X

Let $\mathcal{D} \in \mathcal{F}$ be the domain of a function

$$f : \mathcal{D} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}.$$

In Royden, μ = Lebesgue measure m, on the measurable sets $\mathcal{M}$.
In Strichartz, $\mathcal{F}$ usually equals the Borel sets $\mathcal{B}$.

<u>Def</u> (Theorem)
Then f is <u>measurable</u> iff any one of the following 5 equivalent conditions hold

(i) $\{x \in \mathcal{D} \text{ s.t. } f(x) > \alpha\} = f^{-1}((\alpha, \infty]) \in \mathcal{F}$ $(\forall \alpha \in \mathbb{R})$

(ii) $f^{-1}([\alpha, \infty]) \in \mathcal{F}$ $(\forall \alpha \in \mathbb{R})$

(iii) $f^{-1}([-\infty, \alpha)) \in \mathcal{F}$ $(\forall \alpha \in \mathbb{R})$

(iv) $f^{-1}([-\infty, \alpha]) \in \mathcal{F}$ $(\forall \alpha \in \mathbb{R})$

(v) $f^{-1}(\mathcal{O}) \in \mathcal{F}$ $\forall \mathcal{O} \subset \mathbb{R}$ open, and $f^{-1}(\{\infty\})$, $f^{-1}(\{-\infty\})$ are measurable.

These statements imply

(vi) $\{x \mid f(x) = \alpha\} = f^{-1}(\{\alpha\})$ is measurable $(\forall \alpha \in \overline{\mathbb{R}})$

<u>Proof</u>:

(i) $\Rightarrow$ (ii): $\displaystyle f^{-1}([\alpha, \infty]) = \bigcap_{n=1}^{\infty} f^{-1}((\alpha - \tfrac{1}{n}, \infty])$ ∎ !

can you do the rest?

(ii) $\Rightarrow$ (i): $\displaystyle f^{-1}((\alpha, \infty]) = \bigcup_{n=1}^{\infty} f^{-1}([\alpha + \tfrac{1}{n}, \infty])$ ∎ !

<u>Theorem</u> Let $(X, \mathcal{F}, \mu)$ be a measure space

$\mathcal{D} \in \mathcal{F}$

$f, g : \mathcal{D} \to \mathbb{R}$ measurable

$c \in \mathbb{R}$

(you will do the case $f, g : \mathcal{D} \to \overline{\mathbb{R}}$ in your HW)

Then

(i) $f + c$

(ii) cf

(iii) $f + g$

(iv) $f - g$

(v) fg

(vi) $1/g$ (if e.g. $\frac{1}{g(x)} := +\infty$ $\forall x$ s.t. $g(x) = 0$.)

are also measurable

<u>proof</u>

(i) $(f+c)^{-1}((\alpha, \infty]) = \{x \mid f(x) + c \in (\alpha, \infty]\}$

$= \{x \mid f(x) \in (\alpha - c, \infty]\}$

$= f^{-1}((\alpha - c, \infty]) \in \mathcal{F}$ ∎

(ii) If $c > 0$, $(cf)^{-1}((\alpha, \infty]) = \{x \mid \alpha < cf(x) \le \infty\} = \{x \mid \frac{\alpha}{c} < f(x) \le \infty\}$

$= f^{-1}((\frac{\alpha}{c}, \infty]) \in \mathcal{F}$

If $c < 0$, $(cf)^{-1}((\alpha, \infty])$

$= \{x \mid \alpha < cf(x) \le \infty\}$

$= \{x \mid \frac{\alpha}{c} > f(x) \ge -\infty\} = f^{-1}([-\infty, \frac{\alpha}{c})) \in \mathcal{F}$

If $c = 0$, $(cf)^{-1}((a, \infty]) = \begin{cases} \emptyset & a \ge 0 \\ \mathcal{D} & a < 0 \end{cases}$ each in $\mathcal{F}$.

(iii) clever! Consider

$A = (f+g)^{-1}((a, \infty]) = \{x \mid \alpha < f(x) + g(x) \le \infty\}$.

$= \bigcup_{q \in \mathbb{Q}} \{x \mid f(x) > q\} \cap \{x \mid g(x) > \alpha - q\}$!

$A \supset \bigcup_{q \in \mathbb{Q}}$, because each of $(\{x \mid f(x) > q\} \cap \{x \mid g(x) > \alpha - q\}) \subset A$.

Conversely, if $x \in A$, then

$f(x) > \alpha - g(x)$

so $\exists\, q$ s.t. $f(x) > q > \alpha - g(x)$!

Thus A is a countable union of sets in the σ-algebra $\mathcal{F}$, so $A \in \mathcal{F}$. ∎

(iv) implied by (ii), (iii)

(v) Clever! : If $F: \mathcal{D} \to \overline{\mathbb{R}}$ is measurable then so is F^2, since

$$\{x \mid F^2(x) > \alpha\} = \begin{cases} \mathcal{D}, \text{ if } \alpha < 0 \\ \mathcal{D} \setminus F^{-1}(0) \quad \text{if } \alpha = 0 \\ F^{-1}((\sqrt{\alpha}, \infty]) \cup F^{-1}(-\sqrt{\alpha}, -\infty]) \quad \text{if } \alpha > 0. \end{cases}$$

$\therefore \quad \frac{1}{4}\left((f+g)^2 - (f-g)^2\right) = fg$ is measurable, by (ii) - (iv)

(vi) If $\alpha > 0$, $\left(\frac{1}{g}\right)^{-1}((\alpha, \infty]) = g^{-1}([0, \frac{1}{\alpha})) \in \mathcal{F}$

$\alpha = 0$, $\left(\frac{1}{g}\right)^{-1}(0, \infty]) = g^{-1}([0, \infty)) \in \mathcal{F}$

$\alpha < 0$, $\left(\frac{1}{g}\right)^{-1}([-\infty, \alpha)) = g^{-1}((-\infty, \frac{1}{\alpha})) \in \mathcal{F}$

Convergence and measurable functions

Let $(X, \mathcal{F}, \mu)$ be a measure space,

$\mathcal{D} \in \mathcal{F}$, $f: \mathcal{D} \to \mathbb{R}$ (or $\overline{\mathbb{R}}$).

Recall,

<u>Def</u> $\{f_k\} \to f$ <u>uniformly</u>:

(good for sequences of continuous functions

stronger than

<u>Def</u> $\{f_k\} \to f$ <u>pointwise</u> $(\forall x \in \mathcal{D}, \forall \varepsilon > 0 \ \exists\, m$ s.t. $k \geq m \Rightarrow |f_k(x) - f(x)| < \varepsilon\}$

stronger than

<u>Def</u> $\{f_k\} \to f$ <u>almost everywhere</u> (a.e.)

means $\exists\, E \in \mathcal{F}, \mu(E) = 0$ s.t. $\forall x \in \mathcal{D} \setminus E$, $\{f_k(x)\} \to f(x)$ pointwise

(example: if $f_k(x) \to f(x)$ $\forall x \notin C$, the cantor set)

Related concept:

Def

Let $f, g : \mathcal{D} \to \overline{\mathbb{R}}$.

Then $f = g$ a.e. means $\exists \underset{\mu(E)=0}{E \in \tilde{\mathcal{F}}}$ s.t. $f(x) = g(x) \quad \forall x \in \mathcal{D} \setminus E$.

Theorem: If the measure space $(X, \tilde{\mathcal{F}}, \mu)$ is such that μ is an outer measure on X and $\tilde{\mathcal{F}}$ is the σ-algebra of measurable subsets of X (as with $m = m^*$ in Royden) then

A) Every measure zero set $E \subset X$, $\mu(E) = 0$ is in $\tilde{\mathcal{F}}$.

B) If $f, g : \mathcal{D} \to \overline{\mathbb{R}}$, if f is measurable, and if $f = g$ a.e. then g is also measurable.

proof. A) Let $A \subset X$. Need to show $\mu(A) \geq \mu(A \cap E) + \mu(A \sim E)$

But, $\mu(A) \geq \mu(A \sim E)$
$= \mu(A \sim E) + \underbrace{\mu(A \cap E)}_{=0}$ ∎

B) Let $f = g \quad \forall x \in \mathcal{D} \setminus E$.

Then $g^{-1}((\alpha, \infty]) = \underbrace{\left[f^{-1}((\alpha, \infty]) \cap (\mathcal{D} \setminus E) \right]}_{\text{measurable}} \cup \underbrace{\left[g^{-1}((\alpha, \infty]) \cap E \right]}_{\text{measure zero}}$

$\in \tilde{\mathcal{F}}$ ∎

<u>Theorem</u>: Let $(X, \tilde{\mathcal{F}}, \mu)$ be a measure space

$\mathcal{D} \in \tilde{\mathcal{F}}$.

$\{f_k\} : \mathcal{D} \to \overline{\mathbb{R}}$ measurable.

Then so are

(i) $\inf_k f_k$ $\qquad\qquad (\inf_k f_k)(x) := \inf_k \{f_k(x)\}$

(ii) $\sup_k f_k$

(iii) $\limsup_{k\to\infty} f_k$

(iv) $\liminf_{k\to\infty} f_k$

If $\tilde{\mathcal{F}}$ is the collection of measurable subsets for (what began as an outer) measure μ, then all measure zero subsets of X are measurable, and therefore

(v) if $\{f_k\} \to f$ pointwise a.e.
then f is measurable.

<u>Remark</u>: This theorem shows that the concept of <u>measurable function</u> is superior to that of <u>continuous function</u>, if one is considering pointwise (or pointwise a.e.) convergence.

proof: $(\inf_k f_k)(x) \geq \alpha$ iff $f_k(x) \geq \alpha \ \forall \alpha$

(i) so $(\inf_k f_k)^{-1}([\alpha, \infty]) = \bigcap_k f_k^{-1}([\alpha,\infty]) \in \tilde{\mathcal{F}}$. ∎

(ii)

(iii)

(iv) $(\liminf_{k\to\infty} f_k)(x) = \liminf_{k\to\infty} f_k(x) = \lim_{n\to\infty}\left(\inf_{k\geq n} f_k(x)\right)$ is the limit of an increasing sequence,

so $\liminf_{k\to\infty} f_k(x) \leq \alpha$ iff $\inf_{k\geq n} f_k(x) \leq \alpha \ \forall n$ ∎

(v): $f = \liminf_{k\to\infty} f_k$ a.e. ∎