HW for Friday
April 17
Royden p 69-70 # 22, 24, 25, 28
p 86 # 3, 4, 7, 8

Math 5210
Friday April 10

Question day on measures.

Hopefully you now have some appreciation of the different ways (e.g. metric outer measures) of "measuring" sets.

At the end of the day, we want to end up with a σ-algebra $\mathcal{A}$, and a <u>measure</u> μ, i.e. s.t. $\mu(E) \in [0,\infty]$ $\forall E \in \mathcal{A}$, and

$$\mu(\emptyset) = 0$$
$$\mu(\bigcup E_k) = \sum_k \mu(E_k) \quad \text{for all ctble disjoint collections } \{E_k\} \subset \mathcal{A}.$$

On Monday we return to Royden (§3.5, measurable functions), so that we can develop Lebesgue integration in Chapter 4. (You can also read §14.3 Strichartz, which is more condensed.)

(For Royden, the measure μ is Lebesgue measure m, restricted to $\mathcal{M}$, but most of the discussion is unchanged for a general measure μ on a σ-algebra $\mathcal{A}$ of subsets of a space X.)

<u>Def</u> $(X, \mathcal{A}, \mu)$ is a <u>measure space</u> if X is a space, $\mathcal{A}$ is a σ-algebra, and μ is a measure defined on $\mathcal{A}$.

<u>Def</u>. Let $(X, \mathcal{A}, \mu)$ be a measure space, and $f : X \to \overline{\mathbb{R}}$ $(= \mathbb{R} \cup \{+\infty, -\infty\})$.
Then f is <u>measurable</u> (w.r.t. μ) iff

$$f^{-1}(\mathcal{O}) \in \mathcal{A} \quad \forall\, \mathcal{O} \subset \overline{\mathbb{R}},\ \mathcal{O} \text{ open}$$

<u>Remark</u> If $X = (X, d)$ and $\mathcal{A}$ contains the Borel σ-algebra $\mathcal{B}$, then all continuous functions are measurable, since f cont $\Leftrightarrow$ $f^{-1}(\mathcal{O})$ open $\forall \mathcal{O}$ open.

<u>Theorem</u> (c.f. p.65 Royden, for $\mu = m$, $\mathcal{A} = \mathcal{M}$). f is measurable iff any one of the following are true:

i) $\forall \alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty]) \in \mathcal{A}$
ii) $\forall \alpha \in \mathbb{R}$, $f^{-1}([\alpha, \infty]) \in \mathcal{A}$
iii) $\forall \alpha \in \mathbb{R}$, $f^{-1}([-\infty, \alpha)) \in \mathcal{A}$
iv) $\forall \alpha \in \mathbb{R}$, $f^{-1}([-\infty, \alpha]) \in \mathcal{A}$

Furthermore, f is measurable $\Rightarrow$ $f^{-1}(\{z\}) \in \mathcal{A}$, $\forall z \in \overline{\mathbb{R}}$.