Math 5210
Wednesday April 1.

Measurable sets & Borel sets
$\mathcal{M}$ $\qquad$ $\mathcal{B}$

postpone the two Strichartz HW problems - so for this Friday just complete
Royden p. 56 $\quad$ 6, 7, 8
p. 62 $\quad$ 10, 11, 13, 14b

- Finish Thm page 3 Tuesday. Since the proof there forgot to show that a countable union of msble sets is msble, and since there was confusion in class, let's start over. We already know $\mathcal{M}$ is a Boolean algebra.

Theorem

A) Let $\{E_n\} \subset \mathcal{M}$, $\{E_n\}$ disjoint.
Let $A \subset \mathbb{R}$.

Then $$m^*\left(A \cap \left(\bigcup_{n=1}^{\infty} E_n\right)\right) = \sum_{n=1}^{\infty} m^*(A \cap E_n)$$

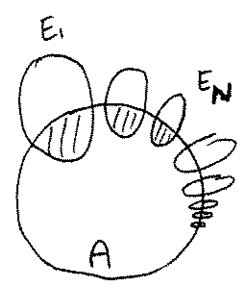
In particular, $$m^*\left(\bigcup_n E_n\right) = \sum_{n=1}^{\infty} m^*(E_n)$$ countable additivity

B) Let $\{F_n\} \subset \mathcal{M}$. Then $\bigcup_{n=1}^{\infty} F_n \in \mathcal{M}$.
Thus $\mathcal{M}$ is a σ-algebra

proof Let $\{E_n\} \subset \mathcal{M}$, $\{E_n\}$ disjoint.
Since $\mathcal{M}$ is a Boolean algebra, the partial union
$$\bigcup_{n=1}^{N} E_n \in \mathcal{M}.$$
Thus, for any $A \in \mathbb{R}$,

this is just analogous to how you use partial sums to understand infinite series.

(1) $$\boxed{m^*(A) = m^*\left(A \cap \left(\bigcup_{n=1}^{N} E_n\right)\right) + m^*\left(A \setminus \left(\bigcup_{n=1}^{N} E_n\right)\right)}$$



To prove A), we expand the first term on the right: Because E_N is measurable,
$$\underbrace{m^*\left(A \cap \left(\bigcup_{n=1}^{N} E_n\right)\right)}_{B} = \underbrace{m^*(A \cap E_N)}_{B \cap E_N} + \underbrace{m^*\left(A \cap \left(\bigcup_{n=1}^{N-1} E_n\right)\right)}_{B \cap E_N^c}$$

So by induction,

(2) $$\boxed{m^*\left(A \cap \left(\bigcup_{n=1}^{N} E_n\right)\right) = \sum_{n=1}^{N} m^*(A \cap E_n)}$$

yesterday I tried (and failed) to save time by using our work from the Boolean algebra theorem.

Thus $\quad m^*\left(A \cap \left(\bigcup_{n=1}^{\infty} E_n\right)\right) \geq m^*\left(A \cap \left(\bigcup_{n=1}^{N} E_n\right)\right) = \sum_{n=1}^{N} m^*(A \cap E_n).$

$\lim_{N\to\infty} \Rightarrow \quad \boxed{m^*\left(A \cap \left(\bigcup_{n=1}^{\infty} E_n\right)\right) \geq \sum_{n=1}^{\infty} m^*(A \cap E_n)}$

this proves (A), since $\boxed{m^*\left(\bigcup_{n=1}^{\infty} (A \cap E_n)\right) \leq \sum_{n=1}^{\infty} m^*(A \cap E_n)}$

by subadditivity. ■

To prove B), let $\{F_n\} \subset \mathcal{M}$

Then $E_1 := F_1 \in \mathcal{M}$

$E_n := F_n \setminus \left(\bigcup_{j=1}^{n} F_j\right) \in \mathcal{M}$

and $\{E_n\} \subset \mathcal{M}$, disjoint, and $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$

so we show $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$.

If $\{F_n\}$ had been an increasing union this is analogous to ~~[illegible]~~ expressing an increasing sequence as a positive term series.

Combining (1), (2) yields

$$m^*(A) = \sum_{n=1}^{N} m^*(A \cap E_n) + m^*\left(A \setminus \bigcup_{n=1}^{N} E_n\right)$$

subadditivity $\Rightarrow$

$$m^*(A) \geq \sum_{n=1}^{N} m^*(A \cap E_n) + m^*\left(A \setminus \bigcup_{n=1}^{\infty} E_n\right)$$

$\lim_{N\to\infty} \Rightarrow$

$$m^*(A) \geq \sum_{n=1}^{\infty} m^*(A \cap E_n) + m^*\left(A \setminus \bigcup_{n=1}^{\infty} E_n\right)$$

subadditivity $\Rightarrow$

$$\boxed{m^*(A) \geq m^*\left(\bigcup_{n=1}^{\infty}(A \cap E_n)\right) + m^*\left(A \setminus \bigcup_{n=1}^{\infty} E_n\right)}$$

$$= m^*\left(A \cap \left(\bigcup_{n=1}^{\infty} E_n\right)\right) + m^*\left(A \setminus \bigcup_{n=1}^{\infty} E_n\right)$$

This proves $\bigcup_{n=1}^{\infty} E_n$ is measurable (B) ■

Def $E \subset \mathcal{M}$, write $m(E) := m^*(E)$, and call $m(E)$ the "measure" of E
(the word outer has disappeared.)

So, if I is an interval, $I \in \mathcal{M}$ and

$m(I) = \ell(I)$

$m(I+y) = m(I)$ ← you prove this in HW

$m(\bigcup E_n) = \sum m(E_n)$ if $\{E_n\} \subset \mathcal{M}$ and collection disjoint ← This also implies that if $\{E_n\}$ is an increasing union, $E_n \subset E_{n+1}\ \forall n$, then $m(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} E_n$.

$\mathcal{O}$ open $\Rightarrow \mathcal{O} \subset \mathcal{M}$

pf: this is because $\mathbb{R}$ is separable:

Let $\mathcal{O}$ open. Then

$$\mathcal{O} = \bigcup_{\substack{p,q \in \mathbb{Q} \\ (p,q) \subset \mathcal{O}}} (p,q)$$

proof: $\supset$ follows by def.

$\subset$: let $x \in \mathcal{O}$. $\Rightarrow \exists\, r>0$ s.t. $(x-r, x+r) \subset \mathcal{O}$
pick $p \in \mathbb{Q}$, $x-r<p<x$
$q \in \mathbb{Q}$ $x<q<x+r$
$\Rightarrow (p,q) \subset \mathcal{O}$. ∎

Thus $\mathcal{O} \in \mathcal{M}$.

so also closed sets are in $\mathcal{M}$

also $G_\delta := \{E \subset \mathbb{R}$ s.t. $E = \bigcap_{n\in\mathbb{N}} \mathcal{O}_n$, $\mathcal{O}_n$ open$\}$ ctble $\cap$'s of open sets.

$F_\sigma = \{E \subset \mathbb{R}$ s.t. $E = \bigcup_n F_n$, F_n closed$\}$.

$G_{\delta\sigma}$ = ctble unions of G_δ sets

⋮

e.g. the cantor set is an intersection of closed sets, so is closed. But if we'd have thrown away closed subintervals instead of open ones we'd have made a G_δ set.

Def The Borel subsets of $\mathbb{R}$ ($\mathcal{B}$) is the smallest σ-algebra containing the open sets.

↑ $\exists$, since the condition $E \in \mathcal{B}$ iff $E \in$ every σ-algebra containing all open sets defines a σ-algebra!

$\mathcal{B} \subset \mathcal{M}$

You can show $\mathcal{B} \neq \mathcal{M}$.

If you trust the axiom of choice you can prove $\mathcal{M} \neq 2^{\mathbb{R}} = \{$all subsets of $\mathbb{R}\}$.

Example **Cantor Set**

$C_0 = [0,1]$

$C_1 = [0,\tfrac{1}{3}] \cup [\tfrac{2}{3},1]$

$C_2 = [0,\tfrac{1}{9}] \cup [\tfrac{2}{9},\tfrac{1}{3}] \cup [\tfrac{2}{3},\tfrac{7}{9}] \cup [\tfrac{8}{9},1]$

$\vdots$

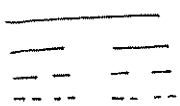
(delete middle thirds)

$C = \bigcap_k C_k$ is closed.

$\Rightarrow m^*(C_k) = 2^k\left(\frac{1}{3^k}\right) = \left(\frac{2}{3}\right)^k$

$C \subset C_k$

$\Rightarrow m^*(C) \leq m^*(C_k) = \left(\frac{2}{3}\right)^k \to 0$ as $k \to \infty$.



so C has measure 0.
on the other hand, C is of the same cardinality of $[0,1]$.

sketch: $x \in [0,1] \Rightarrow x$ has an (almost unique) expression base 3,

$$x = .a_1 a_2 a_3 \cdots = \sum_{j=1}^{\infty} a_j 3^{-j}, \quad a_j \in \{0,1,2\}.$$

unique if eliminate ∞ tails of zeroes.

$x \notin C_1$ iff $a_1 = 1$

$x \notin C_2$ iff $x \notin C_1$ and $a_2 \neq 1$

$\vdots$

$\boxed{x \in C \text{ iff no } a_j = 1.}$

But also if $x \in [0,1]$ we may write

$$x = \sum_{j=1}^{\infty} b_j 2^{-j} \qquad b_j \in \{0,1\},$$ unique if eliminate ∞ tails of 0's.

$f : [0,1] \to C$ is a bijection!
$b_j \longmapsto 2b_j$

If you had used open intervals for the C_j, then the decreasing intersection would've been a G_δ set.

You can play games with how much length you delete at each stage, to get sets of arbitrary Lebesgue measure (between 0 and 1), see HW.