Math 5210

4th hw assignment, due 2/11

Class Exercise 6

9.2.5 # 9, 11, 13

9.3.7 # 1-3, 6, 10

Class exercise 6:

Show (1) $\mathring{A} \cup \mathring{B} \subset (A \cup B)^{\circ}$ (3) $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$

(2) $\mathring{A} \cap \mathring{B} = (A \cap B)^{\circ}$ (4) $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$

give examples in (1), (3) to show sets may not be equal.

<u>Lemma</u> $X \subset Y \Rightarrow \mathring{X} \subset \mathring{Y}$, $\bar{X} \subset \bar{Y}$

<u>pf</u>: if $x \in \mathring{X}$ $\exists B_r(x) \subset X \subset Y$ so $x \in \mathring{Y}$

pf if $x \in X$ or is a limit point of X then $x \in Y$ or is a limit point of Y!

(1): $A \subset A \cup B \Rightarrow \mathring{A} \subset (A \cup B)^{\circ}$ $B \subset A \cup B \Rightarrow \mathring{B} \subset (A \cup B)^{\circ}$

so $\mathring{A} \cup \mathring{B} \subset (A \cup B)^{\circ}$

<u>Example</u> $A = [1,2]$, $B = [2,3]$ $\mathring{A} \cup \mathring{B} = (1,2) \cup (2,3)$, $(A \cup B)^{\circ} = (1,3)$.

(2): $\mathring{A} \cap \mathring{B} \subset A \cap B$

$\Rightarrow (\mathring{A} \cap \mathring{B})^{\circ} \subset (A \cap B)^{\circ}$

$(\mathring{A} \cap \mathring{B})^{\circ} = \mathring{A} \cap \mathring{B}$ since $\mathring{A} \cap \mathring{B}$ is open and the interior of an open set is the set itself.

so $\boxed{\mathring{A} \cap \mathring{B} \subset (A \cap B)^{\circ}}$

but $\left.\begin{array}{l} A \supset A \cap B \\ B \supset A \cap B \end{array}\right\} \Rightarrow \left.\begin{array}{l} \mathring{A} \supset (A \cap B)^{\circ} \\ \mathring{B} \supset (A \cap B)^{\circ} \end{array}\right\} \Rightarrow \boxed{\mathring{A} \cap \mathring{B} \supset (A \cap B)^{\circ}}$

the 2 boxes $\Rightarrow \mathring{A} \cap \mathring{B} = (A \cap B)^{\circ}$ ∎

alternate proof:

let $x \in \mathring{A} \cap \mathring{B} \Rightarrow \exists r_1, r_2$ $B_{r_1}(x) \subset A$, $B_{r_2}(x) \subset B$

$r := \min(r_1, r_2) \Rightarrow B_r(x) \subset A \cap B$

$\Rightarrow x \in (A \cap B)^{\circ}$

conversely, let $x \in (A \cap B)^{\circ} \Rightarrow \exists B_r(x) \subset A \cap B$

$\Rightarrow B_r(x) \subset A$ and $B_r(x) \subset B$

$\Rightarrow x \in \mathring{A}$ and $x \in \mathring{B}$

$\Rightarrow x \in (\mathring{A} \cap \mathring{B})$. ∎

(3) $A \cap B \subset \bar{A} \cap \bar{B}$

$\Rightarrow \overline{A \cap B} \subset \overline{\bar{A} \cap \bar{B}} = \bar{A} \cap \bar{B}$

since the closure of a closed set is the set itself.

example

$A = (0,1)$, $B = (1,2)$

$A \cap B = \emptyset \Rightarrow \overline{A \cap B} = \emptyset$.

but $\bar{A} \cap \bar{B} = [0,1] \cap [1,2] = \{1\}$.

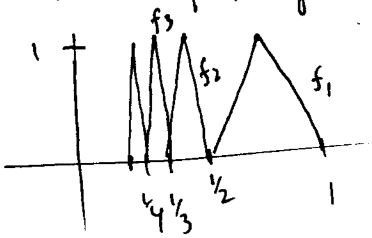
(4) $A \cup B \subset \bar{A} \cup \bar{B}$

$\boxed{\Rightarrow \overline{A \cup B} \subset \overline{\bar{A} \cup \bar{B}} = \bar{A} \cup \bar{B}}$

$\left.\begin{array}{l} A \cup B \supset A \\ A \cup B \supset B \end{array}\right\} \Rightarrow \left.\begin{array}{l} \overline{A \cup B} \supset \bar{A} \\ \overline{A \cup B} \supset \bar{B} \end{array}\right\} \Rightarrow \boxed{\overline{A \cup B} \supset \bar{A} \cup \bar{B}}$

boxes imply $\overline{A \cup B} = \bar{A} \cup \bar{B}$

9.2.5 9. Here's a picture of the graphs of the fcns:



i.e. $f_k(x) = \begin{cases} 0 & x \geq 1/k \\ 0 & x \leq 1/(k+1) \\ 1 & x = \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k+1}\right) \end{cases}$
piecewise linear on the intervals
$\left[\frac{1}{k+1}, \frac{1}{2}\left(\frac{1}{k}+\frac{1}{k+1}\right)\right]$
$\left[\frac{1}{2}\left(\frac{1}{k}+\frac{1}{k+1}\right), \frac{1}{k}\right]$.

$d(f_k, 0) = 1$ since $\max_{x\in[0,1]} |f_k(x) - 0| = 1$.

$d(f_k, f_\ell) = 1$ $k \neq \ell$ since $|f_k(x) - f_\ell(x)| \leq 1$
and equality is attained at exactly 2 x-values.

11. Pick $x_k \in A_k$. Then $\{x_k\}_{k \geq m} \subset A_m$.

Now A_1 compact $\Rightarrow \{x_k\}$ has convergent subseq. $\{x_{k_j}\} \to x \in A_1$
but $k_j \geq m \Rightarrow x_{k_j} \in A_m$ compact
so $x \in A_m \ \forall m$
$\Rightarrow x \in \bigcap_k A_k$ ∎

13. $\mathbb{Q}$ is dense in $\mathbb{R}$, and countable.
the product of countable sets is countable (see text §1.2)
thus $\mathbb{Q}\times\mathbb{Q} \subset \mathbb{R}^2$ is ctble
$(\mathbb{Q}\times\mathbb{Q})\times\mathbb{Q} \subset \mathbb{R}^3$ "
and by induction, $\mathbb{Q}^n \subset \mathbb{R}^n$ is countable.
It is also dense because for $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ and $\varepsilon > 0$
pick $q_i \in \mathbb{Q}$, $|q_i - x_i| < \frac{\varepsilon}{\sqrt{n}}$ $i = 1, \ldots n$
then $\|x - q\|_2 < \left(\sum_{i=1}^n \frac{\varepsilon^2}{n}\right)^{1/2} = \varepsilon$.
$|q_i - x_i|^2 < \frac{\varepsilon^2}{n}$.

9.3.7
1. $f: M \to N$ continuous
* iff $f^{-1}(\mathcal{O})$ open $\forall \mathcal{O} \subset N$ open.
but $\mathcal{O}$ open iff $\mathcal{O}^c$ closed
and $f^{-1}(\mathcal{O}^c) = (f^{-1}(\mathcal{O}))^c$ since $f(x) \in \mathcal{O}^c$ iff $f(x) \notin \mathcal{O}$!
so
* $\Leftrightarrow$ $(f^{-1}(\mathcal{O}))^c$ closed $\forall$ $\mathcal{O}^c \subset N$ closed
 $\| \qquad\qquad\qquad\qquad \|$
$f^{-1}(\mathcal{O}^c)$ closed $\forall$ $K \subset N$ closed
 $\|$
$f^{-1}(K)$ ∎

2. $d(x, x_0): M \to \mathbb{R}$ cont: let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Then $d(x,y) < \delta$
$x \in M$
reverse Δ ineq Δ ineq
$\Rightarrow$ $d(x,x_0) - d(x,y) \leq d(y, x_0) \leq d(y,x) + d(x,x_0)$
$< \delta + d(x, x_0)$
$= \varepsilon + d(x, x_0)$
$d(x,x_0) - \varepsilon = d(x, x_0) - \delta < d(x,x_0) - d(x,y)$ ~~$\varepsilon = d(x,x_0) - \varepsilon$~~

2. cont'd. i.e. $d(x,x_0) - \varepsilon < d(y,x_0) < d(x,x_0) + \varepsilon$

$$-\varepsilon < d(y,x_0) - d(x,x_0) < \varepsilon$$

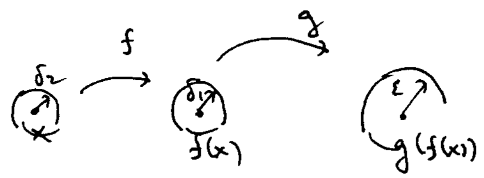
$$|d(y,x_0) - d(x,x_0)| < \varepsilon \quad \blacksquare$$

3. $f : M \to N$ cont
$g : N \to P$ cont $\Rightarrow g \circ f : M \to P$ cont.

proof (1) balls: let $\varepsilon > 0$, $x \in M$.

g cont $\Rightarrow \exists\, \delta_1$ s.t. $g(B_{\delta_1}(f(x))) \subset B_\varepsilon(g(f(x)))$

f cont $\Rightarrow \exists\, \delta_2$ s.t. $f(B_{\delta_2}(x)) \subset B_{\delta_1}(f(x))$



$$\Rightarrow g(f(B_{\delta_2}(x))) \subset g(B_{\delta_1}(f(x))) \subset B_\varepsilon(g(f(x))) \quad \blacksquare$$

or

(2) sequences: let $\{x_k\} \to x$ in M

f cont $\Rightarrow \{f(x_k)\} \to f(x)$ in N

g cont $\Rightarrow \{g(f(x_k))\} \to g(f(x))$ in P $\blacksquare$

or

(3) open sets: let $\mathcal{O} \subset P$ open

$\to g^{-1}(\mathcal{O}) \subset N$ open, g cont

$\Rightarrow f^{-1} g^{-1}(\mathcal{O}) \subset M$ open f cont

$$f^{-1} g^{-1}(\mathcal{O}) = (g \circ f)^{-1}(\mathcal{O}). \quad \blacksquare$$

(you could've used any one of these proofs)

6. ~~sets/balls~~: (1) Balls:

$f : M \to \mathbb{R}^n$

$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$ is cont iff each $f_i(x)$ is, $i = 1, 2, \dots n$

$\Rightarrow$: Let f cont., $x \in M$, $\varepsilon > 0$ $\Rightarrow \exists\, \delta > 0$ s.t. $d(y,x) < \delta \Rightarrow \|f(x) - f(y)\|_2 < \varepsilon$

$$|f_j(x) - f_j(y)| \le \left(\sum_{i=1}^n (f_i(x) - f_i(y))^2 \right)^{1/2} = \|f(x) - f(y)\|_2$$

any $1 \le j \le n$.

$\Rightarrow$ each f_j continuous.

$\Leftarrow$: Let $x \in M$, $\varepsilon > 0$. pick δ_i s.t. $d(y,x) < \delta_i \Rightarrow |f_i(x) - f_i(y)| < \frac{\varepsilon}{\sqrt{n}}$, $i = 1, \dots n$

Let $\delta = \min(\delta_1, \dots \delta_n)$. Then

$$d(y,x) < \delta \Rightarrow \left(\sum_{i=1}^n |f_i(x) - f_i(y)|^2 \right)^{1/2} < \left(n \varepsilon^2/n \right)^{1/2} = \varepsilon \quad \blacksquare$$

or

(2) sequences.

Let $\{x_k\} \to x$.

Then $\{f(x_k)\} \to f(x)$ in $\mathbb{R}^n$

iff each $\{f_i(x_k)\} \to f_i(x)$ $i = 1 - n$

Thm 9.2.4 page 415.

Thus sequential continuity holds for $f : M \to \mathbb{R}^n$ iff it holds for each f_i. $\blacksquare$

9.3.7 #10). $f_k : M \to N$, $k \in \mathbb{N}$.

Def $\{f_k\} \to f$ uniformly iff $\forall \varepsilon > 0\ \exists\, m$ s.t. $k \geq m \Rightarrow \sup_{x \in M} d(f_k(x), f(x)) < \varepsilon$.

Theorem: If each f_k is continuous
and $\{f_k\} \to f$ uniformly
then f is continuous.

pf Let $x \in M$
$\varepsilon > 0$. Pick m s.t. $k \geq m \Rightarrow \sup_{x \in M} d(f_k(x), f(x)) < \varepsilon/3$

in particular,

$$d(f_m(y), f(y)) < \varepsilon/3 \quad \forall y \in M. \qquad (*)$$

Since f_m is cont (at x), pick $\delta > 0$ s.t.

$$d(x,y) < \delta \Rightarrow d(f_m(x), f_m(y)) < \varepsilon/3. \qquad (**)$$

Then $d(x,y) < \delta$

$$\begin{aligned} \Rightarrow d(f(x), f(y)) &\leq d(f(x), f_m(x)) + d(f_m(x), f_m(y)) + d(f_m(y), f(y)) \\ &< \underset{(*)}{\varepsilon/3} + \underset{(**)}{\varepsilon/3} + \underset{(*)}{\varepsilon/3} \\ &= \varepsilon \quad \blacksquare \end{aligned}$$